Understanding the Optimal Redshift Range for the Supernovae Hubble Diagram

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ABSTRACT

The supernovae Hubble diagram traces the expansion history of the universe, including the influence of dark energy. Its use to probe the cosmological model can fruitfully be guided by heuristic study of the features of the model curves. By relating these directly to the physics of the expansion we can understand simply the optimal redshift range for supernovae observations, requiring a survey depth of $z = 1.5 - 2$ to distinguish evolution in dark energy properties.

Energy Densities and Acceleration

The Friedmann equations describe the physics of the expansion in terms of the energy density in the components, $\{\Omega_i\}$, and their equations of state $\{w_i \equiv p_i/\rho_i\}$:

$$(\dot{a}/a)^2 = H_0^2 \left[\sum_i \Omega_i'(z) + (1-\Omega_T)(1+z)^2\right] \tag{1}$$

$$\ddot{a}/a = -(1/2)H_0^2 \sum_i (1+3w_i)\Omega_i'(z), \tag{2}$$

with $\Omega_T = \sum_i \Omega_i = \sum_i \Omega_i'(0)$ and the energy densities of independent components evolving according to

$$\Omega_i'(z) \equiv (8\pi/3H_0^2)\rho_i(z) = \Omega_i\,(1+z)^3 e^{3\int_0^{\ln(1+z)} d\ln(1+z')\,w(z')}, \tag{3}$$

$$\to \Omega_i\,(1+z)^{3(1+w)} \qquad (w\ \text{constant}). \tag{4}$$

Because of these differing evolutions with redshift, the ratio of energy densities changes with time and the dominant dynamic component at one epoch, that driving the expansion rate, may give way to another component. Similarly the acceleration behavior of the expansion may alter, with a different component dominating. The global expansion may even change from deceleration ($\ddot{a} < 0$) to acceleration ($\ddot{a} > 0$), signifying a total equation of state sufficiently negative to break the strong energy condition ($w_T < -1/3$). We denote such switching epochs as z_{eq} and z_{ac} respectively.

The redshift of equality between two components, taken here to be nonrelativistic matter ($w = 0$) and some other component, presumably with negative equation of state, is when $\Omega'_m(z_{eq}) = \Omega'_w(z_{eq})$:

$$z_{eq} = \left(\frac{\Omega_w}{\Omega_m}\right)^{\frac{1}{-3w}} - 1, \tag{5}$$

when w is constant (note Ω's are always present values, unlike $\Omega'(z)$).

The switch to an accelerating universe takes place when $\ddot{a} = 0$, at, assuming only two components are dynamically important,

$$z_{ac} = \left[-(1+3w)\frac{\Omega_w}{\Omega_m}\right]^{\frac{1}{-3w}} - 1, \tag{6}$$

again assuming w constant. The quantities z_{eq} and z_{ac} are plotted in Figure 1 for two flat models, $\Omega_m + \Omega_w = 1$. The results do not greatly change if we allow varying w or nonflat models.

Most of the action in the changeover of the dynamics, both for the dominant energy density component and the expansion acceleration, happens at fairly low redshifts, $z \approx 0.5$, for reasonable values of the matter density. This is in marked contrast to distinguishing between cosmological models with fixed equation of state, say between models with different matter density, where the expansion behavior diverges at high redshifts. Does this mean that to detect the main influence of a dark energy component with $w < -1/3$ we need only probe the expansion history to this redshift? The answer is no, as would be seen

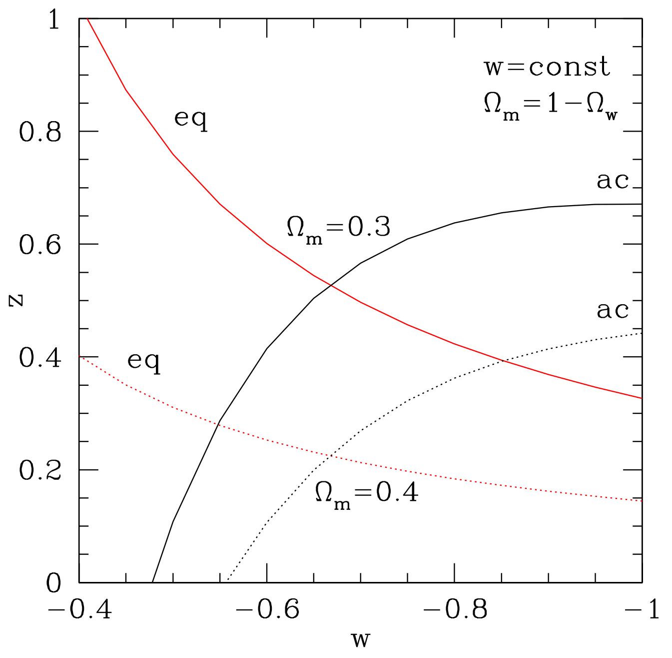

FIG. 1. The redshifts of matter-dark energy equality and of the transition from decelerating to accelerating expansion are plotted vs. equation of state of the dark energy, for a flat universe. The solid curves have $\Omega_m = 0.3$, the dotted $\Omega_m = 0.4$.

from Monte Carlo simulations of the cosmological parameter likelihood determinations, but here we shall come to this conclusion using a more transparent, intuitive examination of the physics behind the expansion history.

Also note that a cosmological model can have $z_{eq} < z_{ac}$ or $z_{eq} > z_{ac}$, i.e. first start accelerating then become w dominated or v.v., depending on the equation of state. So the equation of state parameter, of the dark energy or the effective total energy, is a key variable and one that can be probed sensitively through the supernovae Hubble diagram out to moderate redshifts.

In the following sections we examine the redshift range for determination of the equation of state, discussing use of the effective total equation of state, evolving $w(z)$ and "inertia" or "memory" in the magnitude-redshift diagram, and discrimination between models including maximum deviation and turnover rate characteristics.

Hubble Diagram

The two quantities entering the Hubble diagram – magnitude and redshift – are direct tracers for the expansion history of the universe. The magnitude is a logarithmic measure of the luminosity distance, related to the lookback time – dimmer means further in the past, while the redshift is exactly the relative expansion since that time; thus $m(z)$ is $a(t)$ in a transparent sense. Curvature in the diagram probes evolution in the expansion rate: acceleration, directly dependent on the equation of state as seen from eq. (2).

Explicitly, the relations are

$$
\begin{aligned}
m(z) &\sim 5\log r_l(z), && (7)\\
r_l(z) &= (1+z)(1-\Omega_T)^{-1/2}\sinh\left[(1-\Omega_T)^{1/2}\int_0^z dz'\,[H(z')/H_0]^{-1}\right] && (8)\\
&\rightarrow (1+z)\int_0^z dz'\,[H(z')/H_0]^{-1},\\
H(z)/H_0 &= [\Omega'_m(z)+\Omega'_w(z)+(1-\Omega_T)(1+z)^2]^{1/2} && (9)\\
&\rightarrow (1+z)^{3/2}[\Omega_m+\Omega_w e^{3\int_0^{\ln(1+z)} d\ln(1+z')\,w(z')}]^{1/2},
\end{aligned}
$$

where the restricted cases are for flat universes, $\Omega_T=\Omega_m+\Omega_w=1$.

At the lowest redshifts, the luminosity distance depends on parameters as

$$
r_l = z + \frac{1}{2}(1-q_0)z^2 + z^3 f(\Omega_T,\langle w\rangle,\langle w^2\rangle), \qquad (10)
$$

where angle brackets denote density weighted averaging, $\langle w\rangle = \sum_i w_i\Omega_i/\Omega_T$; note $\langle w\rangle = w_T$, the effective equation of state of all the components combined.

The magnitude-redshift test therefore probes not only the present acceleration, q_0, but differing combinations of cosmological parameters in different redshift ranges. This is one of its great strengths in that not only is it most sensitive to changes, e.g. acceleration, in the expansion history of the universe at recent times, but it provides 1) complementarity to high redshift tests, other medium redshift tests, and even itself over different redshift ranges; 2) detection of evolution in the state and nature of the universe by probing different epochs than the cosmic microwave background, say, and different energy densities (smooth components) than large scale structure tests; 3) protection against secular or differently evolving systematic effects such as grey dust.

To examine its ability to discriminate between cosmological models first consider models with a single component but different values for the magnitude of the energy density, e.g. different Ω_m. The difference in magnitudes between such models is monotonically increasing with redshift, until $r_l \sim \Omega_m^{-1}z$ at asymptotically high redshifts, so discrimination improves with ever increasing supernovae redshift (observational difficulties aside).

But for a competing component, especially one with negative pressure, its influence is seen over a limited redshift range and little advantage accrues to a deep survey – discrimination is squeezed in between the common linear distance-redshift law at small redshifts and the matter dominated relation at high redshifts. The question addressed here is whether the optimal redshift range is near $z = 0.5$ or higher, and how broad the window is.

Effective Equation of State

First consider a series of flat, constant w models with fixed $\Omega_w = 1 - \Omega_m$. One might expect that their magnitude differentials $\Delta m_{w'} = m_w - m_{w'}$ also diverge with increasing redshift, and that the optimal depth is therefore pushed (at least formally) to high redshift. But this is not quite true as seen in Figure 2.

Models with fixed equation of state but different proportions of dark energy and matter indeed diverge, showing increasing magnitude differences with redshift as illustrated by the dashed curve. But models with fixed energy densities and differing equations of state (constant in time) level off and no great advantage accrues to probing the cosmology with supernovae at higher redshift. A clear signature of models with both differing dark energy density and equation of state is the nonmonotonic behavior of the dotted curve, also a property of evolving equations of state as we will discuss. These different characteristics

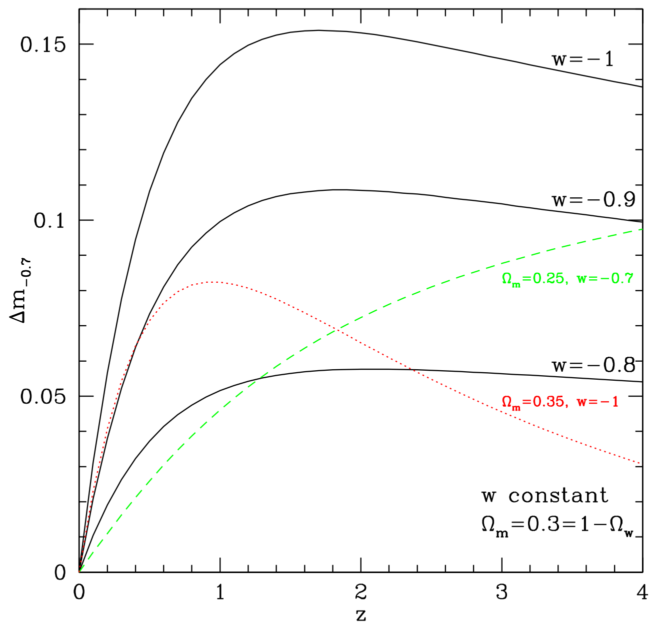

FIG. 2. The differential magnitude-redshift relation is plotted for various equations of state relative to the $w = -0.7$ case. The solid curves have constant w (and fixed Ω_w) but do not diverge with redshift the way curves of Ω_w (at fixed w) would (compare the dashed curve), due to dilution by a decelerating component. For comparison the dotted curve shows the turnover behavior upon varying both w and Ω_w.

of the shape, or curvature, of the magnitude-redshift relation thus provide important clues to the cosmological model.

The curvature in the differential Hubble diagram $\Delta m - z$ probes the differential acceleration between the models, which receives contributions from both the negative pressure component and the increasingly dominant matter. While this relation between curvature and acceleration is obvious at low redshift, as seen in equation (10), the physical correspondence continues at higher redshifts as shown by the following argument.

Consider r_l; the key cosmological variable determining r_l is $H^{-1}(z)$. (It is actually the integral over redshift that enters, which will lead to an "inertia" effect discussed later.) Whether models continue to diverge or turn around depends on the evolution of H^{-1}. The "scale height" of this evolution is

$$\frac{d\ln H^{-1}}{d\ln(1+z)} = -\frac{dH^{-1}}{dt} = -(1+q), \tag{11}$$

so the evolution in the differential Hubble diagram is indeed due to the differential acceleration of the models. This in turn can be related to the effective, or total equation of state by $\Delta q \propto \Delta w_T$ since

$$q(z) = (1/2)\sum_i (1+3w_i)\Omega_i'(z) \Big/ \left[\sum_i \Omega_i'(z) + (1-\Omega_T)(1+z)^2\right], \tag{12}$$

and the total equation of state

$$\begin{aligned} w_T \equiv p_T/\rho_T &= \sum_i w_i\Omega_i'(z) / \sum_i \Omega_i'(z) \\ &\to w\Omega_w'(z)/[\Omega_m'(z)+\Omega_w'(z)] = w \Big/ \left[1+\frac{\Omega_m}{\Omega_w}e^{-3\int d\ln(1+z)\, w}\right], \end{aligned} \tag{13}$$

with the restriction for the matter plus dark energy model and $w_T(0) = w\Omega_w/\Omega_T$.

Thus the key observational characteristic of curvature in the Hubble diagram is a direct probe of the total equation of state of the universe. Note that even for constant w the presence of a matter component as well ensures that w_T evolves with redshift, with the negative pressure gradually being diluted to zero at high redshift. From eq. (13) the dilution is obviously enhanced for larger Ω_m/Ω_w. An interesting property is that the dilution can be nonlinear, since initially more negative equations of state are diluted more rapidly due to their diminishing energy density at higher redshifts, and so curves of w_T can cross: $w_T(w) > w_T(w')$ at high redshift despite $w < w'$.

Evolving w models

The same expression, eq. (13), holds for defining w_T in the case of evolving dark energy. Since w_T, through differential acceleration, forms the physical basis for understanding the deviation and curvature in the magnitude-redshift diagram, figures like Figure 3 are

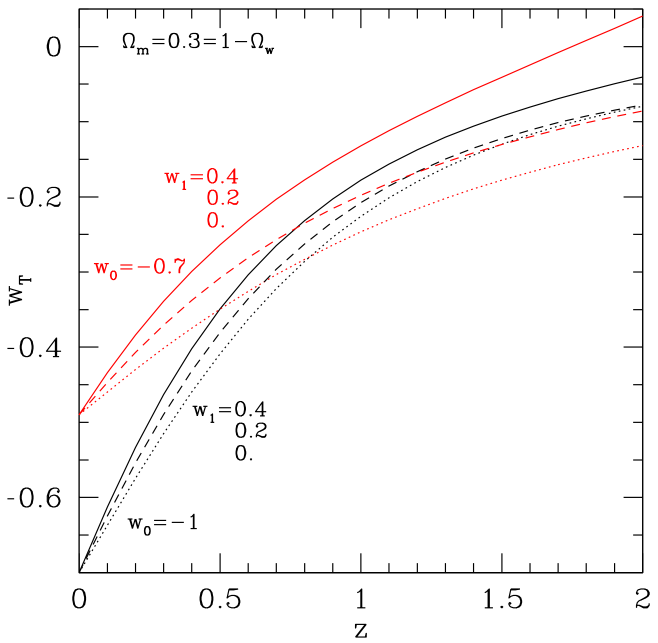

FIG. 3. The total or effective equation of state is plotted vs. redshift. Dotted curves are constant w models, others have evolving $w = w_0 + w_1 z$, with $w_1 = 0.2$ for dashed curves, $w_1 = 0.4$ for solid curves. All models are flat. Even constant w models effectively evolve with redshift due to matter dilution, approaching $w = 0$. Dilution is nonlinear with a model containing a more negative pressure component overtaking a less negative one and becoming more matterlike (decelerating).

invaluable for an intuitive grasp of how models behave and how cosmological parameters can be distinguished as a function of survey depth.

Here we see that constant w models are diluted and do indeed cross, with the effective equation of state of the cosmological constant ($w = -1$) model actually greater (more matterlike and decelerating of the expansion) than the $w = -0.7$ model at $z \geq 0.8$. Evolving models are linearly parametrized, $w = w_0 + w_1 z$. Positive w_1 of course increases the dilution, and if sustained for long enough can drive w and w_T positive, but one must be careful not to extend the linear evolution approximation beyond its validity.

Using the w_T's of two models as guides to their differential acceleration, one can deduce the behavior of their differential magnitude-redshift behavior $\Delta m(z)$. One expects that a model with w_T less than the fiducial model (hence greater acceleration of the expansion) would have a positive magnitude offset at a given redshift, i.e. possess apparently dimmer supernovae. This offset would increase with redshift until the greater dilution of w_T for the more negative w model causes it to approach the w_T value of the fiducial. On the magnitude diagram this would appear as a slowing of the deviation and eventually a turnover at approximately the redshift when the two w_T's agree (but see the Inertia section later). If the effective equations of state then keep pace, the turnover is just a leveling off, while if the w_T curves cross then the magnitude deviation approaches zero (and can go negative though eventually it levels off at redshifts fully matter dominated).

For example, from Figure 3 we would predict a priori that models (w_0, w_1) should have the following behaviors in their (differential) Hubble diagrams:

- (-0.7,0.4)-(-0.7,0): increasingly negative magnitude deviation with no turnover or slowing.
- (-0.7,0.2)-(-0.7,0): increasingly negative magnitude deviation with no turnover but gradual slowing.
- (-1,0.2)-(-1,0): increasingly negative magnitude deviation with no turnover but leveling off.
- (-0.7,0.2)-(-1,0): increasingly negative magnitude deviation with leveling beyond $z = 1.5$ and very gradual turnover.
- -[(-1,0.4)-(-0.7,0)]: increasingly negative magnitude deviation with leveling beyond $z = 0.5$ and turnover.

The corresponding Hubble diagrams are shown in Figure 4, bearing out these heuristic predictions.

Thus knowledge of the equation of state w for a model enables direct qualitative construction of its magnitude-redshift relation through w_T, including the sign of the deviation,

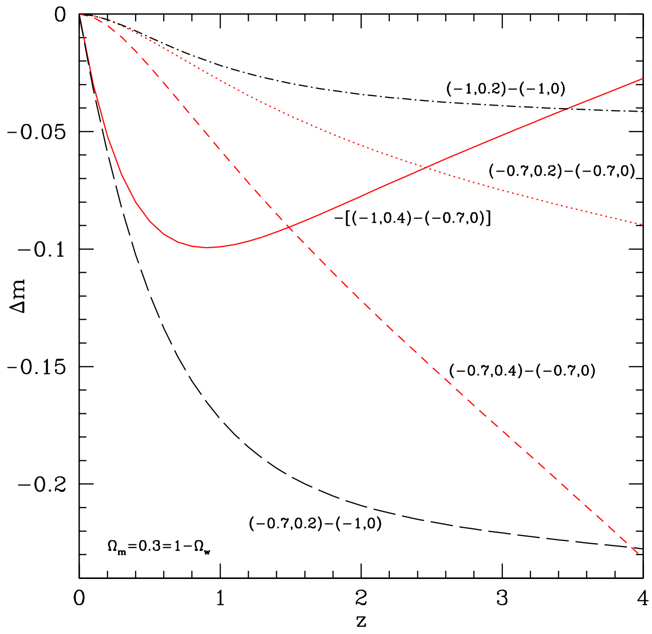

FIG. 4. Differential Hubble diagrams plot the magnitude differences between the $w(z) = w_0 + w_1 z$ models labeled with (w_0, w_1), exhibiting a variety of divergence, leveling, and turnover behaviors. Heuristic analysis by means of the total equations of state is successful in predicting the qualitative features.

the presence of leveling or turnover, and the sharpness of the evolution. Quantitative results such as the size of the maximum magnitude deviation must be gotten by numerical evaluation of the luminosity distance from eq. (8). However we can obtain a rough estimate of the turnover redshift, and hence the required survey depth, by examining the w_T curves as above. This method becomes more definite in the Inertia section.

The inverse process of looking at a differential Hubble diagram and intuitively extracting the model equation of state is more problematic. One could probably get a rough sense of w_T at the redshift of any turnover, and an indication of how rapidly it was increasing at redshifts lower and higher than this, but more quantitative estimates necessitate numerical analysis. The main virtue of the heuristic method though is estimation of the redshift range required to distinguish models, to find the "sweet spot" of the Hubble diagram that discriminates between different constant w models and between constant and evolving w models. The key to this is the turnover from w_T representing the differential acceleration.

Inertia

While the criterion on w_T for the turnover works reasonably and was justified physically through the differential acceleration and scale length argument of the Effective Equation of State section, it is not the instantaneous evolution of the Hubble parameter $H(z)$ that governs the distance and magnitude, but rather its behavior over the entire redshift range between the source and observer. This memory (or foreshadowing) of $w(< z)$ produces an inertia in the magnitude-redshift relation at z so that it does not respond immediately to $w_T(z)$.

For example, even though w_T for two models may have converged, the inertia of the integral over z keeps them from having a purely constant offset in magnitude Δm. Consider constant flat models with $w = -1, 0.7$. By $z = 2$ (5;10) their $w_T = -0.08, -0.132$ (-0.011,-0.036; -0.002,-0.010) yet their $\Delta m = 0.15$ (0.13, 0.11), i.e. the offset is neither negligible, as it would be if only $w_T(z)$ was important, nor strictly constant, as it would be if the region where $\Delta w_T(z) \approx 0$ gave no contribution to Δm. Rather the $\Delta w_T(z) \approx 0$ region adds a constant contribution to each model's distance r_l which leads to a steady dilution of $\Delta m \sim \log(r_l/r_l')$.

Inertia is also responsible for the turnover occurring at higher redshifts than the w_T crossing. While this cannot be made rigorously quantitative with a solely intuitive, rather than numerical, approach the w_T analysis does provide a good rough estimate (and a lower limit) for the redshift range critical for distinguishing the influence of dark energy in the Hubble diagram.

One can see how well this approximation works in the numerical results of Figure 5, for both constant and evolving equations of state. While w_T predicts turnovers at $z = (0.8, 0.5, 0.35)$ and (0.96,0.6,0.34) in the $w_0 = -1$ and -0.8 cases respectively for increasing w_1, they occur at $z = (1.7, 0.9, 0.6)$ and (>2,1.13,0.55). (Note that asking where the w's rather than w_T's cross is more accurate because dilution acts opposite to inertia, but it is less physically justified and is undefined in the case of constant w models.)

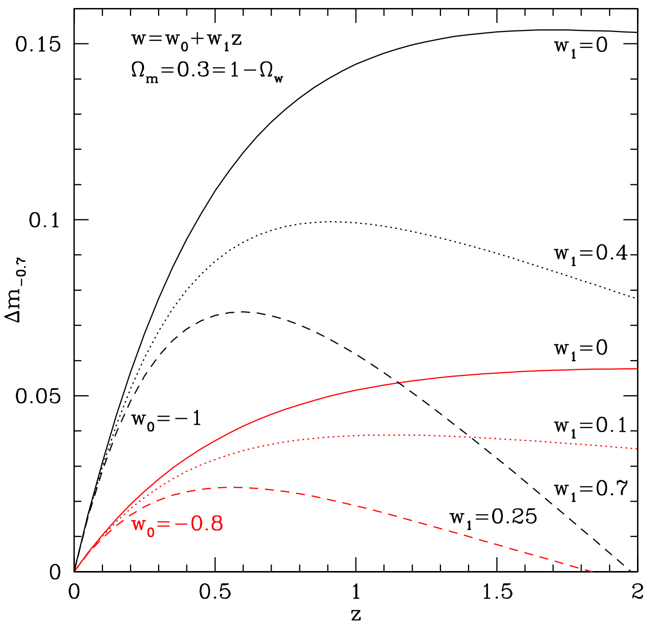

FIG. 5. Differential magnitude-redshift relations are plotted for flat, linearly evolving dark energy models $w = w_0 + w_1 z$ relative to the constant model $w = -0.7$. Prospective SNAP (SuperNova/Acceleration Probe) error bars of 0.02 in magnitude will be able to distinguish between constant and evolving dark energy and also between sufficiently different evolution behaviors by observing in redshift out to $z = 1.7$.

Two interesting points are that for all six models the w_T criterion defines a redshift about half of the redshift at peak magnitude deviation (turnover) and the magnitude deviation there is consistently within 11-13% of the peak deviation, due to the gradualness of the leveling or turnover. If these characteristics were shown to be fairly general (as they seem to be given the wide range of w_0 and w_1 in the six models) then the w_T criterion could in fact be calibrated to be a fairly robust predictor of both the turnover redshift and magnitude deviation.

Summary

The Hubble diagram of Type Ia supernovae is well suited to probing the cosmological model for the existence and properties of dark energy. Such a component is expected to dominate the energy density and dynamics, most evident in the acceleration of the expansion, at redshifts of $z \leq 0.5 - 1$. Thus at these moderate redshifts these models can clearly be distinguished from pure matter, or other positive energy, models. However to discriminate *between* dark energy models, whether between cosmological constant, constant w tracking models, or evolving quintessence models, it is necessary to extend the survey depth to $z \approx 1 - 2$. This reveals 1) the physical imprint of differential acceleration, 2) the dilution of the effective equation of state due to the increasing dominance of matter, and 3) the turnover to a decelerating expansion including the "inertia" of the magnitude-distance relation that integrates over the equation of state from the source epoch to the present.

This is shown, along with a cautionary note, in Figure 6. Although all the models graphed have their "action" redshifts z_{eq} and z_{ac} at $z < 0.7$, observations extending only to $z = 0.7$ are clearly insufficient to probe the cosmological model. One could not tell whether one is dealing with a constant equation of state different from a cosmological constant, a rapidly evolving dark energy model, or a purely cosmological constant model at a slightly different energy density. Surveys extending out to $z \approx 1.5 - 2$ can make such distinctions, even down to fairly fine differences (the curves were chosen to represent roughly those "confusion limits"). One caution however is that complementary constraints, e.g. on Ω_m, Ω_T or the low redshift behavior, from other cosmological probes or low redshift supernovae data, play a crucial role in limiting the parameter space of alternate models.

For the parameter ranges of interest the general rules of thumb in finding the "sweet spot" of the optimum redshift depth for discrimination between models, especially distinguishing between evolving $w(z)$ and a constant w_c model, are that the redshift range contributions are:

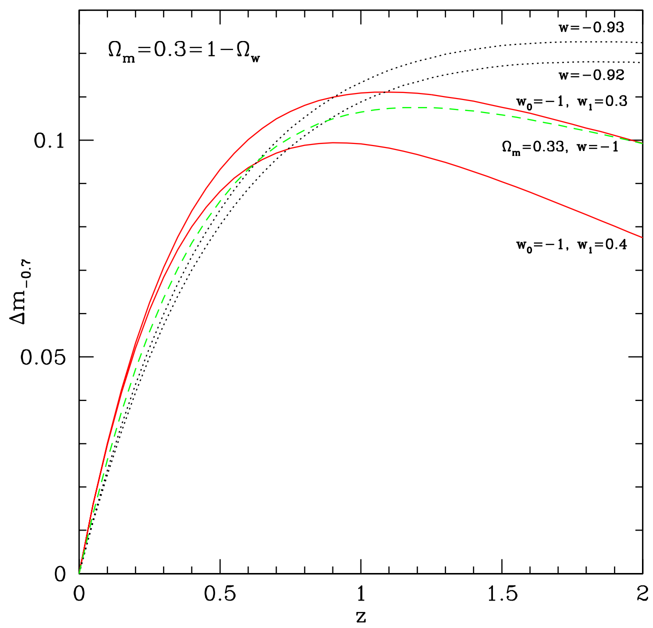

FIG. 6. A "confusion plot" of Hubble diagrams is shown with nearly degenerate low redshift behavior. Although at $z > 0.7$ all these models are matter dominated and decelerating, it is only in this redshift region that this cosmological probe becomes useful, able to distinguish different dark energy equations of state and constant from evolving w. Note however that large uncertainty in other variables such as Ω_m can erode the model parameter determination.

- significant out to z such that $w(z) \approx w_c$, i.e. similar w_T's
  (as the magnitude curves diverge);
- mild at higher z as w_T's become similar
  (differential curve levels off but redshift baseline increases);
- negligible at higher z if the w_T's strongly cross
  (turnover in differential magnitude; note dilution ensures the w_T's never diverge sufficiently at high z to give a large magnitude difference of the reversed sign).

Due to the deceleration dilution effect therefore, for the models considered here the most likely optimal depth for a SNAP supernovae survey is $z \approx 1.5 - 2$.