References

Saha, Prasenjit; Stadel, Joachim; Tremaine, Scott, 1997, A Parallel Integration Method for Solar System Dynamics, Astronomical Journal v.114, p.409

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Mixed Variable Symplectic integrators

For the solar system, the Hamiltonian can be written:

$$H = H_K + H_P$$

where H_K describes the Kepler motion of the planets, and H_P describes the perturbations of the planets on each other.
Now

$$H_K = H_K(\mathbf{J})$$

where $\mathbf{J}$ are the actions of the Kepler potential and

$$H_P = H_P(\mathbf{x})$$

where $\mathbf{x}$ are the cartesian coordinates of the planets. Using operator splitting, we arrive at the leapfrog-like scheme:

1. Move all planets along their Kepler orbits for $1/2$ timestep.
2. Apply perturbation acceleration, keeping $\mathbf{x}$ fixed.
3. Move all planets along their (new) Kepler orbits for $1/2$ timestep.

If H_P is smaller than H_K by a factor ϵ then the integrator is accurate to $O(\epsilon\tau^2)$.

Warmup

The leading error term is from $\mathbf{J} \neq \mathbf{J}_{num}$ to $O(\epsilon)$. The frequencies are unequal, and we get a linear drift in the angles.
We can eliminate this error by adiabatically moving from the "Physical orbit" to the numerical orbit.

1. Integrate backward with a small timestep ($\tau_b < \sqrt{\epsilon}\tau$), while linearly reducing the interaction.

2. Switch to actual timestep.

3. Integrate forward while linearly increasing the interaction to full strength.

Other techniques for eliminating leading error:

1. Symplectic corrector: analytically calculate the transformation.

2. Iterated start: find the point in the orbit where the true orbit and numerical orbit cross.

Integrator is now $O(\epsilon^2\tau^2)$.

A parallel integration method

Consider $\dot{z} = f(z)$. This can be solved with:

$$z_n = z_0 + \tau \sum_{m=0}^{n-1} f\left(\frac{1}{2}(z_m + z_{m+1})\right); \quad n = 1..N$$

This is a set of non-linear equations in $N \times dimen(z)$ variables which we can solve via iteration.

For a Hamiltonian system with $z \equiv (p, q)$ this is an "implicit midpoint" integrator:

$$\begin{aligned} p_1 &= p_0 - \tau \frac{\partial}{\partial q} H\left(\frac{1}{2}(z_0 + z_1)\right), \\ q_1 &= q_0 + \tau \frac{\partial}{\partial p} H\left(\frac{1}{2}(z_0 + z_1)\right) \end{aligned}$$

which is symplectic.

For H of the form

$$H = H_K(p) + H_P(p, q, t)$$

where p, q are action-angles, these become:

$$\begin{aligned} p_n &= p_0 - \tau \sum_{m=0}^{n-1} \frac{\partial}{\partial q} H_P\left(\frac{1}{2}(z_m + z_{m+1})\right), \\ q_n &= q_0 + \tau \sum_{m=0}^{n-1} \left[\frac{\partial}{\partial p} H_K(p_m + p_{m+1}) \right. \\ &\qquad \left. + \frac{\partial}{\partial p} H_P\left(\frac{1}{2}(z_m + z_{m+1})\right)\right]. \end{aligned}$$

This can be implemented as a parallel `SCAN` operation.