Monte Carlo & model realization: Comments and solutions

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$\boxed{1}$ Constructing King models

When trying to do a problem like this, I always go back to the original definitions rather than rely on a book or paper; otherwise, I invariably find that I've misunderstood something. My advice is to use the quoted results as guidance only. In this case, the main pitfalls are variable definition and scaling. Here is my solution ...

1 Setting it up

We want to integrate the Poisson equation:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}V = \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 4\pi G\rho(r).$$

In order to get $\rho(r)$, we integrate the definition for the King distribution function over all velocity at fixed value for r:

$$\begin{aligned}
\rho(r) &= \int d^3v f_o(e^{-E/\sigma^2} - 1) \\
&= 4\pi f_o\left\{e^{-V(r)/\sigma^2}\int_0^{v_{max}} dv v^2 e^{-v^2/2\sigma^2} - \frac{2}{3}v_{max}^3\right\}
\end{aligned}$$

where $v_{max} = \sqrt{2(E_t - V(r))}$ is the maximum velocity for a bound star at radius r. For convenience, we can set $E_t = 0$ (more properly speaking, we can absorb the constant into the definition for potential). And following King, we can define $W = -V(r)/\sigma^2$. Using this definition and letting $\eta = v^2/2\sigma^2$ gives:

$$\rho(r) = 4\pi f_o\left\{\sqrt{2}\sigma^3 e^W\int_0^W d\eta\eta^{1/2}e^{-\eta} - \frac{2\sqrt{2}}{3}\sigma^3 W^{3/2}\right\}$$

$$= \left[\frac{8\pi f_o\sqrt{2}\sigma^3 e^{W_o}}{3}\right] e^{W-W_o}\int_0^W d\eta\eta^{3/2}e^{-\eta}$$

where the last equality follows by integrating by parts. Putting this gives the 2nd order ODE we need to solve:

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr} = 4\pi G\rho_o e^{W-W_o}\int_0^W d\eta\eta^{3/2}$$

where ρ_o is the stuff in [ ] in the expression for $\rho(r)$. Those of you who have a reason look King's paper in the future will recognize this form.

2 Solving it

As many of you realized, we can write this as a dimensionless equation by dividing by σ^2 to turn V into W and then scaling away the $4\pi G\rho/\sigma^2$ into the length scale (squared). Many of you chose to use King's definition of r_c so I will do that here. The scaled equation is:

$$\frac{d^2W}{dx^2}+\frac{2}{x}\frac{dW}{dx} = -9e^{W-W_o}\int_0^W d\eta\eta^{3/2}e^{-\eta}.$$

The initial conditions are $W(x=0)=W_0$ and $y(x=0)\equiv dW/dx=0$ for two coupled first order ODEs. The only nuisance is the integral over η. But we can determine this integral at the same time as the solution to the Poisson equation! One only need compute $q(x=0)\equiv\int_0^{W_o} d\eta\eta^{3/2}$ to start and then integrate along $dq/dx=\eta^{3/2}e^{-\eta}dW/dx$ with the initial condition $q=q(x=0)$. I used the Octave `quad` function to do the quadrature for $q(x=0)$. This turns out to be a lot easier than the method that King gives in his paper. Alternatively, one can change variables and integrate by parts to express the integral in terms of `erf()` (as in B&T).

Figure 1 shows relative profiles in density (ρ) and potential (W) for a range of values in W_0.

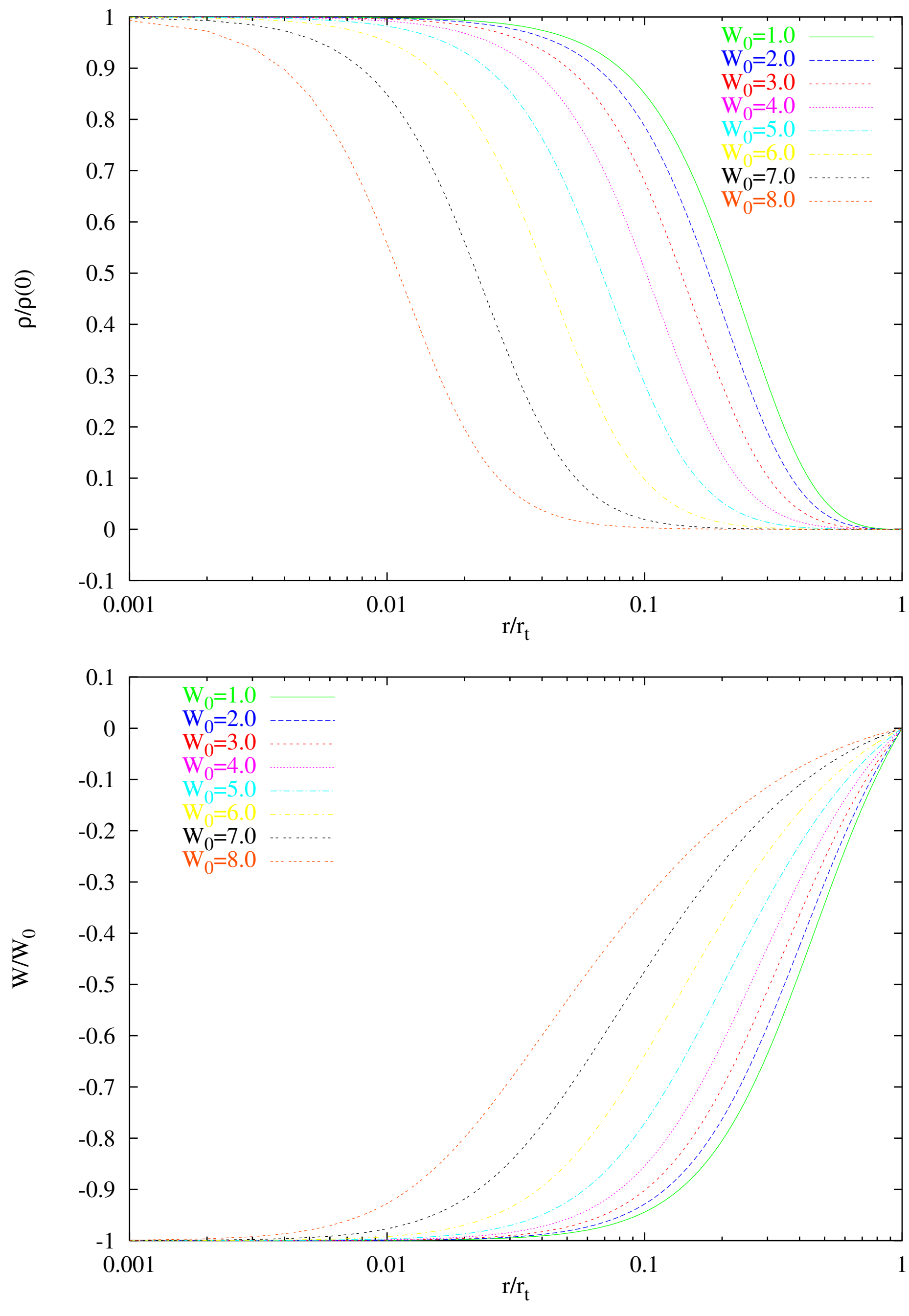

Figure 1: Density (above) and Potential (below) profiles for King models with W_0 as labeled.