Computing the Universe '99:
Monte Carlo realization of distributions

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This is Part 1 of a two part problem set. Part 2 will be posted separately. The problems below will require sampling random variates from uniform and Gaussian distributions. We recommend using the `ranlib` package. The source in both C and Fortran is available from www.netlib.org and locally in `/hosts/a/home/weinberg/ranlib`. This is available on each of your machines. Detailed documentation is available in the `doc` directories for each language and includes reference to the algorithms used.

Here is an example of how to use it. Consider the test program `testranlib.c`:

```
#include <stdlib.h>
#include <ranlib.h>

int main()
{
   int i;

   printf("This routine prints 10 uniform and\n");
   printf("normally distributed random variates:\n");
   for (i=0; i<10; i++) {
      printf("%3d  %13.6e  %13.6e\n",
                 i, ranf(), gennor(0.0, 1.0));
   }

   return 0;
}
```

The routine `ranf()` generates random variates from the uniform interval and `gennor(av, std)` generates variates from a Gaussian distribution with mean *av* and standard deviation *std*. The compile sequence is:

```
cc -o tst testranlib.c -lranlibc -lm.
```

Depending on how you've set up your path, you may need:

```
cc -I/usr/local/include -L/usr/local/lib -o tst tes-
tranlib.c -lranlibc -lm.
```

For completeness, here is the same thing in Fortran:

```
      program testranlib

      print *, 'This routine prints 10 uniform and'
      print *, 'normally distributed random variates:'

      do 10 i=1, 10000
      print *, i, ranf(), gennor(0.0, 1.0)
 10   continue

      end
```

The compile sequence is now:

```
g77 -o tst testranlib.f -lranlibf.
```

$\boxed{1}$ **Constructing King models**

Assume that stars in a galaxy have a Maxwellian velocity distribution with velocity dispersion σ. Jeans' theorem implies that the distribution function must be:

$$f(\mathbf{x}, \mathbf{v}) \propto e^{E/\sigma^2},$$

and therefore

$$\rho(r) \propto e^{V(r)/\sigma^2}.$$

In practice, this distribution has infinite extent and mass. To get around this Ivan King proposed truncating the distribution by subtracting a constant:

$$f(\mathbf{x}, \mathbf{v}) \propto e^{E/\sigma^2} - 1.$$

We assume that $f = 0$ if $e^{E/\sigma^2} < 1$. The resulting models are good models for globular clusters, some elliptical galaxies and halos of spiral galaxies. These models are called King models. The edge of the model is called the "tidal radius", r_t. The profile often is described by the dimensionless potential

$$W(r) \equiv \frac{V(r_t) - V(r)}{\sigma^2}.$$

Given the profile, the overall linear size and mass of the profile can be scaled as desired to any set of units.

(a) Note that Poisson's equation is a second order ordinary differential equation just like Newton's equations and therefore can be written as two coupled first order differential equations. Rewrite the Poisson equation as a system of first order ODEs with "physical boundary conditions" (that is: $V(0) =$ constant, $dV(0)/dr = 0$) in spherical coordinates.

(b) Integrate (numerically) Poisson's equation for the King model. By adjusting $W_0 \equiv W(0)$, one gets models with different concentration; you can check against the table below. Plot up the density and the potential just to see what they look like. You will use these later on to generate initial conditions for a simulation. Check your procedure by computing the values in each of the four columns. The core radius of the King model is defined as $r_c = \sqrt{9\sigma^2/4\pi G\rho_o}$. In these units, the second column describes the total energy of the King model, the third column describes the concentration, c, the fourth is the the central density in units of the mean density and the fifth is the mean square particle radius in units of the tidal radius.

$\boxed{2}$ Monte Carlo realization

In dynamics, we often need to perform an integral such as

$$M = \int d\mathbf{p}\, d\mathbf{q}\, f(\mathbf{p}, \mathbf{q}).$$

The value for the integral M represents the total mass if f is the phase space distribution function. In class, we discussed the Monte Carlo method and Monte Carlo integration which is very useful for integrals like this. Recall that the phase-space distribution function tells us the relative probability of observing the point at $(\mathbf{p}, \mathbf{q})$ in some small volume element $d\mathbf{p}\, d\mathbf{q}$. In addition to an integral over phase space, we often want a procedure for picking mass points according to this probability as an initial condition to an n-body simulation.

Table 1: King model parameters

W_o	$E_{tot}/(GM^2/r_t)$	$\log_{10}(r_t/r_c)$	$\rho(0)/(\frac{M}{4\pi r_t^3/3})$	$\langle r^2\rangle/r_t^2$
1.00	-0.64	0.30	3.21e+01	1.51e-01
1.25	-0.65	0.36	3.52e+01	1.46e-01
1.50	-0.67	0.41	3.89e+01	1.41e-01
1.75	-0.68	0.46	4.33e+01	1.36e-01
2.00	-0.70	0.50	4.86e+01	1.31e-01
2.25	-0.72	0.55	5.50e+01	1.26e-01
2.50	-0.74	0.59	6.27e+01	1.21e-01
2.75	-0.76	0.63	7.21e+01	1.16e-01
3.00	-0.78	0.67	8.39e+01	1.11e-01
3.25	-0.81	0.71	9.85e+01	1.05e-01
3.50	-0.84	0.75	1.17e+02	1.00e-01
3.75	-0.87	0.80	1.41e+02	9.52e-02
4.00	-0.91	0.84	1.71e+02	9.02e-02
4.25	-0.94	0.88	2.11e+02	8.53e-02
4.50	-0.99	0.93	2.64e+02	8.05e-02
4.75	-1.04	0.98	3.36e+02	7.59e-02
5.00	-1.09	1.03	4.34e+02	7.15e-02
5.25	-1.15	1.08	5.70e+02	6.74e-02
5.50	-1.21	1.14	7.63e+02	6.35e-02
5.75	-1.29	1.19	1.04e+03	6.00e-02
6.00	-1.37	1.26	1.44e+03	5.67e-02
6.25	-1.45	1.32	2.04e+03	5.38e-02
6.50	-1.55	1.39	2.95e+03	5.14e-02
6.75	-1.64	1.46	4.32e+03	4.93e-02
7.00	-1.74	1.53	6.43e+03	4.77e-02
7.25	-1.84	1.60	9.68e+03	4.66e-02
7.50	-1.94	1.68	1.46e+04	4.62e-02
7.75	-2.02	1.76	2.21e+04	4.62e-02
8.00	-2.08	1.83	3.32e+04	4.66e-02
8.25	-2.12	1.91	4.90e+04	4.75e-02
8.50	-2.13	1.98	7.08e+04	4.86e-02

A one-dimensional example helps motivate a useful approach. Suppose $f(r)$ represents the distribution of some process with r,

$$M = \int_0^\infty dr f(r),$$

and define

$$M(R) = \int_0^R dr f(r).$$

Then, if we have random numbers, $\mathcal{R}$ between 0 and 1, our distribution is the set R_i such that $\mathcal{R} = M(R_i)/M(\infty)$. This becomes obvious once you note that

$$M = \int_0^\infty dr f(r) = \int_0^\infty dr \frac{dM(r)}{dr} = \int_0^M dM.$$

After many choices of random numbers, $\mathcal{R}$, one is uniformly sampling the integrand and therefore recreating the desired distribution.

This has some useful consequences. Suppose it is inconvenient to perform the integral $M(R)$ but we can do in closed form the integral of a function g that looks like f almost everywhere. Then we have:

$$M = \int_0^\infty dr f(r) = \int_0^\infty dr g(r) \frac{f(r)}{g(r)} = \int_0^{\tilde{M}} d\tilde{M}(r) \frac{f(r)}{g(r)}$$

where

$$\tilde{M}(R) = \int_0^R dr g(r).$$

Recalling our original interpretation, the ratio $f(r)/g(r)$ is the relative probability that the value r is in our distribution if we select uniformly from the domain of $\tilde{M}$. This leads immediately to the so-called *rejection algorithm*:

Let $q = \max[f(r)/g(r)]$ over the entire range r. Select a random variable $\mathcal{R}$ between 0 and 1 to get a set R_i such that $\mathcal{R} = \tilde{M}(R_i)/\tilde{M}(\infty)$. The value R_i is then a member of the desired distribution with probability $p = f(R)/g(R)/\max[f(r)/g(r)]$. Generate a new random variate $\mathcal{R}$ and accept R_i if $\mathcal{R} < p$.

Let's now use the rejection algorithm to generate the phase space for a spherical galaxy.

(a) The the phase space integral for a spherical galaxy takes the following form:

$$M = \int d\mathbf{r}\, d\mathbf{v}\, f(E,L)$$

where E is the orbital energy and L is the orbital angular momentum. Argue and show that this may be rewritten as a triple integral:

$$M = 8\pi^2 \int_0^{r_{max}} dr r^2 \int_0^{v_{max}(r)} dv_r \int_0^{\sqrt{v_{max}^2(r)-v_r^2}} dv_t v_t\, f(E,L)$$

where v_r and v_t is the radial and tangential velocities, r_{max} is the edge of the galaxy (may be ∞), $E_{max} = \Phi(r_{max})$ and $v_{max}(r)$ is the maximum velocity of an orbit passing through the radius r.

(b) The basic plan is to apply the rejection algorithm to each of the integrals. Argue that the first integral, the one over r, is really an integral over $4\pi\rho$. The quantity $M(r)$ may then be tabulated and interpolated as necessary.

(c) In order to apply the rejection algorithm to the second two integrals, the integration limits must be the same on every instance (why??). Change variables to do this. I get:

$$M = 8\pi^2 \int_o^{r_{max}} dr r^2 v_{max}^3(r) \int_o^1 dp\,(1-p^2) \int_o^1 dq\, f(E,L)$$

where $v_r = v_{max}(r)p$, $v_t = v_{max}(r)\sqrt{q(1-p^2)}$. The random variates for p and q are then generated by solving $\mathcal{R} = 3(p-p^3/3)/2$ and $\mathcal{R} = q$. The cubic $p - p^3/3$ may be solved by trigonometric substitution, $p = 2\sin(\theta/3)$.

(d) Make sure you agree that the final algorithm is then:

   i. Select a random variate $\mathcal{R}$ and solve for r such that $M(r) = \mathcal{R}$.

   ii. Generate two random variables from two different values of $\mathcal{R}$ such that:

$$\begin{aligned} x &= 2\sin\left(\frac{\sin^{-1}\mathcal{R}_1}{3}\right), \\ y &= \left(1-x^2\right)\mathcal{R}_2. \end{aligned}$$

   iii. Given r, the values of v_r and v_t are then:

$$\begin{aligned} v_r &= v_{max}(r)x \\ v_r &= v_{max}(r)\sqrt{y} \end{aligned}$$

(e) Generate a phase space according to this algorithm and verify that it is correct.

On to Part 2 . . .