ODEs and dynamics:
Comments and solutions

M. Weinberg

These comments are based on both your questions and write-ups. I will make some additional *overall* comments at the Friday roundtable.

NB: some of the plots below are in color and so you might find it helpful to use a PostScript viewer and increase the magnification rather than print out paper copy.

$\boxed{1}$ **Solving ODEs**

As the problem statement hinted and led you to discover, there are two important features of every ODE algorithm:

(a) *Domain of convergence*: step size above which the solutions will be unstable. This step size is problem dependent. Geometrically, it is clear that this behavior should obtain: our 2nd-order equation as exponentially growing, shrinking and oscillatory solutions. Although we want the oscillatory solution, our difference equation ends up with bits of the growing and shrinking solutions.

(b) *Truncation error*: the algorithms we discussed are derived from polynomial or Taylor series approximations which are truncated at some order. Each step has a remainder or truncation error term. In the case of 4th order Runge-Kutta, the error term is $\propto h^5$. Because we integrate over a finite time interval, we need to take $\propto h^{-1}$ steps. This truncation error then accumulates to give $\propto h^4$ global error for the 4th order Runge-Kutta. Other cases are similar.

Specfic comments:

- The *forward-Euler* has no domain of stability for this problem. Because the error per step is always one sided, the exponentially growing solution can not be suppressed. This suggests that if one symmetrized by added 1/2 of a forward scheme and 1/2 of a backward scheme together that stability might be recovered, and this is the case.

- Convergence for the 4th order Runge-Kutta is shown in Figure 1. This analysis is less useful for symplectic integrators which exploit explicit symmetries of a Hamiltonian problem. This motivates the final part of the problem: testing the methods with an arbitrary one-dimension polynomial right hand side.

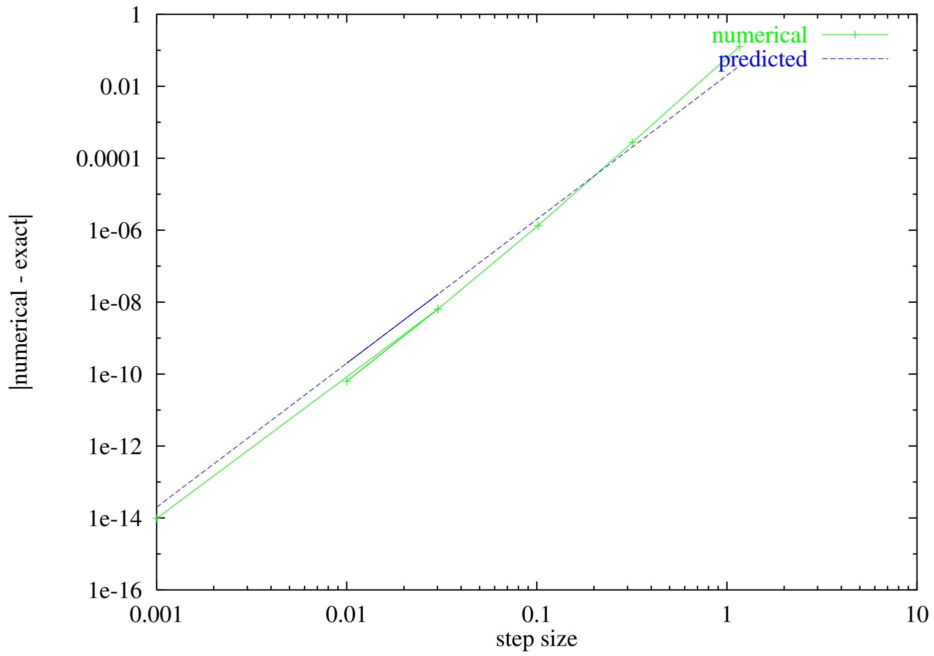

Figure 1: Convergence for 4th order Runge-Kutta. The prediction is error proportional to h^4 (slope 4 line in this log-log plot). Round-off error clobbers this plot for smaller step size.

- One general way of convincing oneself that the orbit integrations are sufficiently accurate is to plot the known conserved quantities as a function of time. Typical will be energy and various projections of angular momentum. This is only a necessary not sufficient check but should *always* be done!

- **IMPORTANT!** One must apply any of these ODE algorithms to each dimension of the coupled ODEs simultaneously (e.g. in parallel) rather than to one variable at a time (e.g. in series). Many folks defined two functions, one for x and one $y = \dot{x}$. A better, more general strategy is to define one function which does both simultaneously.

$\boxed{2}$ **Integrating orbits: a simple case**
See next section.

$\boxed{3}$ **Integrating orbits: a more interesting case**
I will discuss the previous and next closely related problems here. The main point of these problems is to compute both loop and box orbits and identify their features on the corresponding surface of section plot.

Comments:

- Recall: regular orbits are confined by fixed value of one of the coordinate in which Poisson equation separates.
- The only confining coordinate for the axisymmetric case which separates in r and θ is θ. Therefor this problem only has loop orbits.
- The logarithmic potential problem is separable in elliptic coordinates. In this case, there are a variety of combinations of confocal ellipses and hyperbolae which confine orbits. As we discussed in class, these fall into two classes *loop* and *box* orbits.
- The surface of section plot provides a graphical way of distinguishing the two which holds true in higher-dimensional cases where simply plotting the orbit is too difficult: if the surface-of-section curve loops around the origin, the orbit is a box orbit. If the surface-of-section curve avoids the origin, the orbit is a loop orbit.

Figure 2 illustrates this with both a loop and box orbit and their corresponding surfaces of section.

$\boxed{4}$ **Surface of section plots**
See above.

$\boxed{5}$ **Hénon-Heiles potential**
One difficulty here is finding bound orbits. In fact, the energies I suggested all have some bound orbits as you can see by examined the potential:

Some surface of sections and corresponding orbits are shown in the figures[1]. Figures 4–5 show four orbits for $E = 0.05$. The initial conditions four orbits are chosen by a Monte-Carlo procedure. The orbits

[1] These surface-of-section plots contain both positive and negative $\dot{y}$ trajectories; that is I have abandoned the $\dot{y} > 0$ constraint now that we have understood its implications.

and their surface of sections are color coded and line-type coded for comparison (e.g. use a PostScript viewer). The box and loop orbits are easily identified from the surface of section. Note that the surface-of-section tells us that the first orbit is a loop orbit, even though this is not obvious from the plot. Figures 6–7 show the four orbits for $E = 0.1$. The final box orbit is weakly chaotic; note the jitter in the curve especially near the origin. If one increases $E = 0.125$ the surface of section would be clearly volume filling for an analogous orbit (not shown).

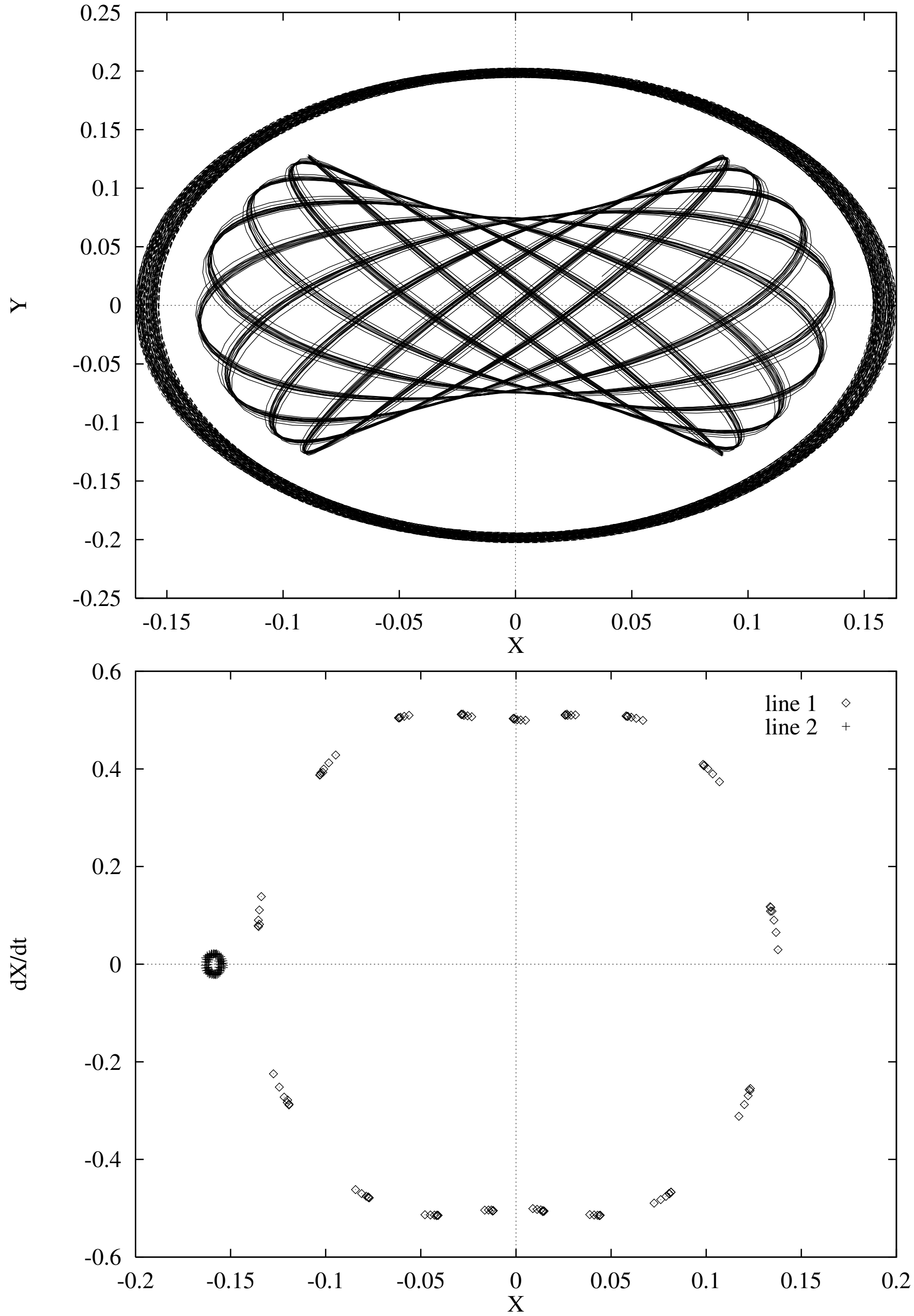

Figure 2: Above: A box [line 1] and (thin!) tube [line 2] orbit. Below: their corresponding surfaces of section. The tube (or loop) orbit is the small circle at the left.

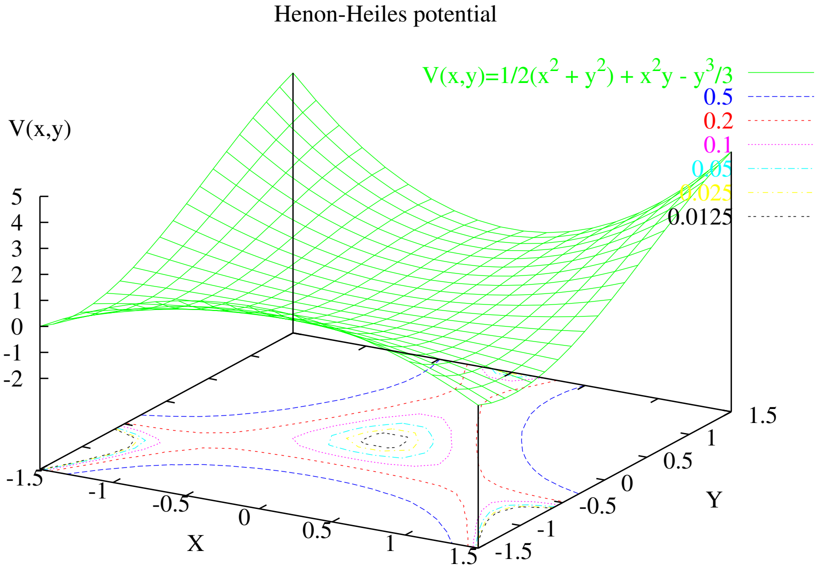

Figure 3: Hénon-Heiles potential. The orbits must be confined to regions inside of $E = V(x,y)$. These curves are sometimes called *zero velocity surfaces*. Note that by starting your solution at large $|x|, |y|$ you can find orbits which are unbound. However, if you start your bound orbits in the central region, they will be confined to the central region.

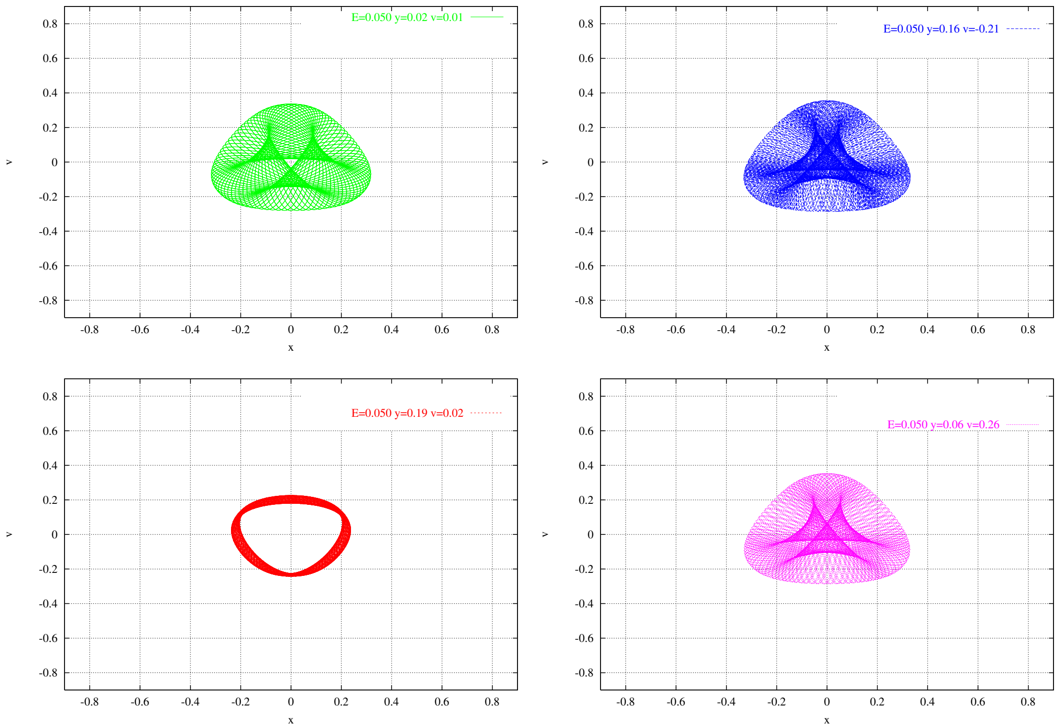

Figure 4: Orbits randomly selected with $E = 0.05$

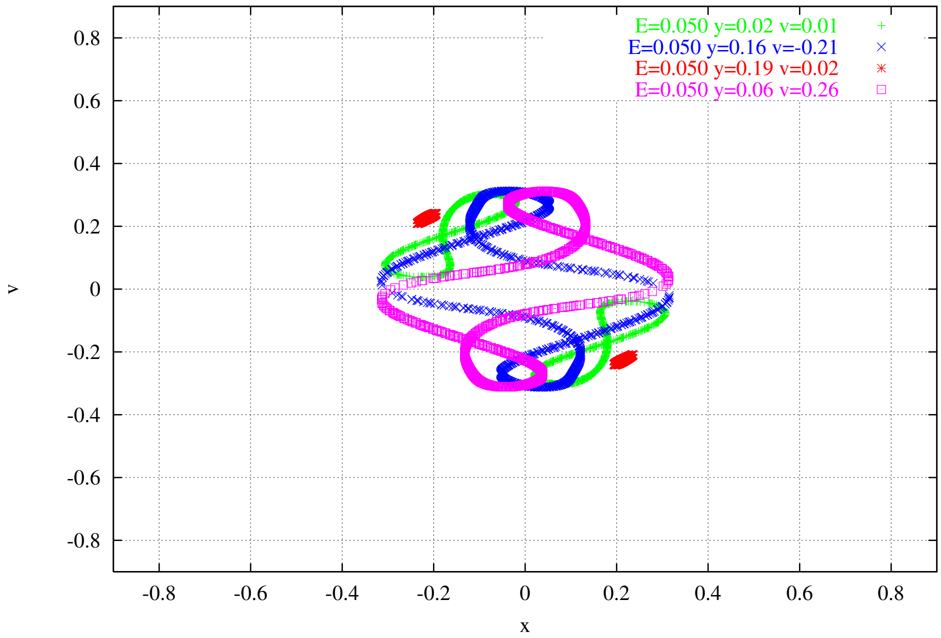

Figure 5: Surface of sections for orbits shown in Fig. 4

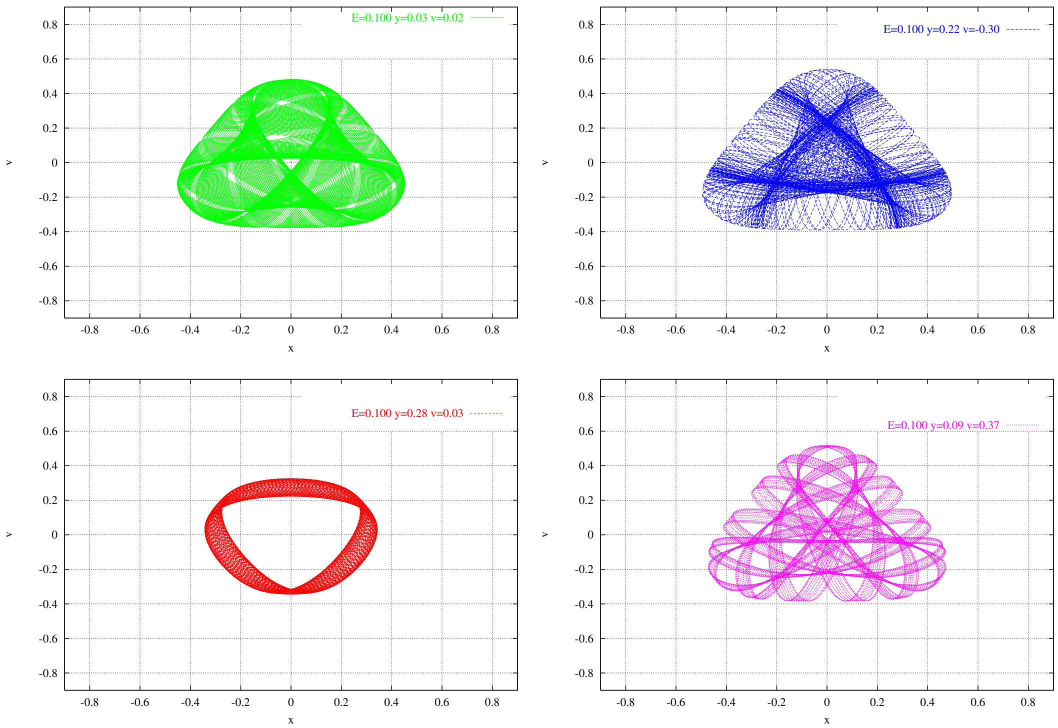

Figure 6: Orbits randomly selected with $E = 0.1$

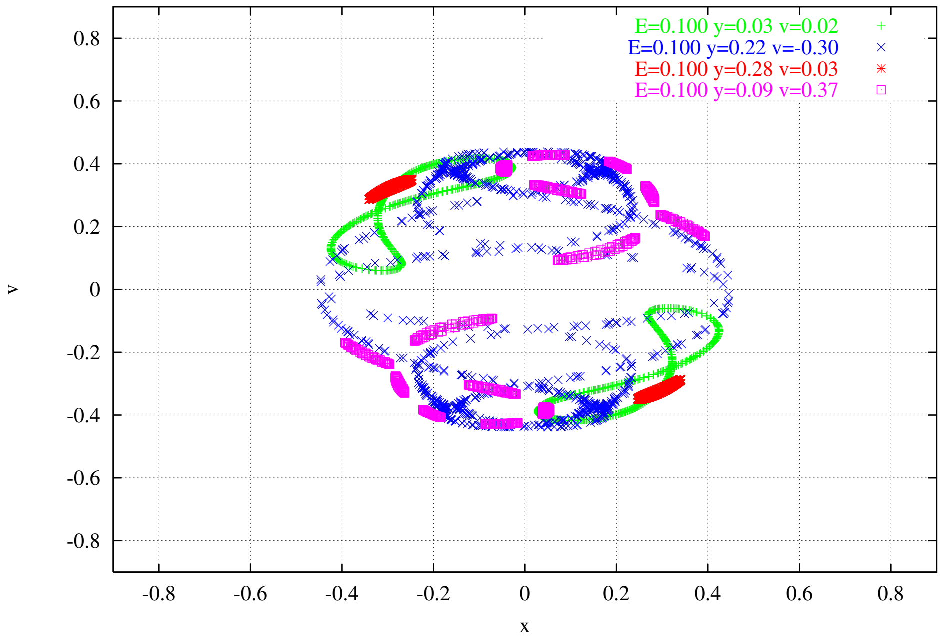

Figure 7: Surface of sections for orbits shown in Fig. 6