Computing the Universe '99:
ODEs and dynamics

June 24, 1999
Due: Tues 29 June

These exercises are designed to give you experience with ODEs and have a bit of fun exploring the orbit morphology and surface-of-section technique described in class. Remember, this is not a race and you don't have to do everything! Take a look at Problems 1-3 which are related and will give you practice with ODE stuff. Then give Problem 4 on surface-of-sections a shot. Problem 5 follows up on Problem 4 and provides an introduction to stochastic orbits. Problem 6 is numerically challenging ... work on this one last and don't worry if you don't get to it.

$\boxed{1}$ **Solving ODEs**

Let's begin by implementing a differential equation solver for an arbitrary number of degrees of freedom. We want to test Forward Euler, Leap Frog and the Runge Kutta algorithms, so it would be easiest to design a code where one can specify the algorithm to use. There are many possible strategies from preprocessing directives to class design (e.g. in C++); it's up to you.

(a) As a test, let's do the phase portrait of harmonic oscillator discussed on Monday:
$$\frac{d^2x}{dt^2} + x = 0, \tag{1}$$
with $x(0) = 1, x'(0) = 1$. This is a good one because can be easily solved analytically.

(b) Solve equation (1) numerically for $0 \geq x \geq 7$ using each of the methods with step sizes 1.6, 0.8, 0.4, 0.2, 0.1, 0.05, 0.025, ...

(c) Each integrator has maximum step size beyond which it is unstable. Roughly locate this step size in each case.

(d) Plot $\log|r_{numerical}(7) - r_{exact}(7)|$ as a function log(step size) and check for the expected convergence of the error term for each algorithm. Explain any discrepancies.

(e) Another good example to check would be 1-d system with polynomial right-hand-sides. For low enough order (figure out this "expected" order before touching the keyboard), the results should have no truncation error. Test this, if you have time.

$\boxed{2}$ Integrating orbits: a simple case

First consider an axisymmetric potential used as a simple model for a galaxian disk first proposed by Toomre (1964, ApJ):

$$U(r) = -(1 + r^2)^{-1/2}.$$

(a) Pick some initial conditions and integrate the orbits. Verify that orbits do not close but precess. These are sometimes called *rosette* orbits because of their shape or *tube* orbits because the have an inner and outer boundary.

(b) Compute the energy and angular momentum of your trial orbits (e.g. using the initial conditions). Compute the inner and outer radii of the tube (the turning points) using the conserved quantities and check these values against your direct orbit integration.

$\boxed{3}$ Integrating orbits: a more interesting case

Consider the non-axisymmetric logarithmic potential:

$$U(x, y) = \frac{1}{2} v_o^2 \log\left(r_c^2 + x^2 + \frac{y^2}{q^2}\right).$$

Let's assume $q = 0.9$, $v_o = 1$ and $r_c = 0.14$.

(a) Pick some initial conditions and integrate the orbits. Note that for very small energies (near the bottom of the potential well), the potential is that of a two-dimensional harmonic oscillator.

(b) Find and identify both loop and box orbits.

$\boxed{4}$ Surface of section plots

Most often, one does not know the fraction of irregular orbits for particular density model. A very useful technique for both determining whether an orbit is regular or irregular and classifying the type of orbit is the "surface of section" method discussed in class.

(a) Adapt your ODE code to find the x—$\dot{x}$ surface of section; that is, have your program plot with points the the values of x and $\dot{x}$ when $y = 0$ and $\dot{y} > 0$.

(b) Use the results from the previous problem to plot the surface of section for both a loop and box orbits. Note/discuss/explain the differences.

(c) Try a few cases with both the Leap Frog and Runga Kutta to convince yourself of the power of a symplectic integrator.

$\boxed{5}$ **Hénon-Heiles potential**
Explore some orbits in the Hénon-Heiles potential:

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2\right) + x^2 y - \frac{1}{3}y^3. \tag{2}$$

(a) Try picking initial conditions with the energies E =0.025, 0.075, 0.125. [I found that these give a nice selection of results but feel free to try others!] For example, one can choose $x = 0$ and y and $\dot{y}$ to get the desired energies.

(b) Compare orbits with surface of section plots at each energy.

$\boxed{6}$ **A numerically ornery orbit integration**
Finally, a more complicated non-axisymmetric case—two fixed point masses—which has the potential:

$$U(x,y) = -\frac{1}{\sqrt{(x-a)^2 + y^2}} - \frac{1}{\sqrt{(x+a)^2 + y^2}} \tag{3}$$

Although far from obvious, this potential can be separated in confocal elliptical coordinates, put in the Hamilton-Jacobi form and therefore has regular orbits. If the two point masses were in orbit around each other, the orbits would not be regular. In other words, the real three body problem has chaotic trajectories!

(a) Again, pick some initial conditions and integrate the orbits. Just to be definite $a = 1/2$ (although the problem scales with a so you could value you wanted).

(b) There are two different kinds of orbits in this potential: *tube* orbits and *box* orbits. Box orbits come arbitrarily close to the center of force filling a closed area (similar to Lissajous figures). In fact, there are two cases of box orbits here: those which come close to one and those which come close to both centers of force. Find an example of each case.

NB: Because box orbits can come arbitrarily close to any point in the "box", you can run into numerical difficulties. One way of getting around this problem is to add a small positive constant ε to the denominator of each in the potential and slowly decrease this to estimate the orbits in the limit $\varepsilon \to 0$.