Computing the Universe, Dynamics III: Actions, orbits and other applications of Hamiltonian dynamics

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1 Isolating and non-isolating integrals of motion

Note that:

- There is no guarantee that we can solve the H-J equation in any particular instance.
- If we can, however, we can reduce the system to multiply periodic motion and solve for the trajectories of all the orbits.
- Conversely, if H-J has no solution, there are missing conserved quantities and the orbits will be *irregular*: particles whose initial conditions are arbitrarily close will diverge exponentially. This is the general case[1].

The actions are examples of *isolating* integrals: they restrict or define location in phase space. The H-J construction shows that every degree of freedom can have at most one action. If all actions

[1] This can be proven formally: the measure of regular orbits are measure zero in the irregular orbits.

can be found, the system is trivial. The solution then looks like the cross of n circles, one for each dimension in the space of actions and angles; topologically, this is an n-torus. In other words, regular motion describes a torus in phase space. (There is a connection here between the actions and the notion that every symmetry has a conserved quantity, discussed briefly on the first day.)

In a number of special cases, one can separate the H-J partial differential equation and find all the actions. For these systems, then, all possible integrals of motion exist. Under what conditions can one do this? Answer: one can show that H-J is separable in any conic coordinate system. These include rectangular, polar, spherical, cylindrical, elliptical, ellipsoidal, spheroidal (two of these: oblate and prolate), hyperbolic (two of these: one sheet and two sheet), parabolic. Of course, there may be special cases that are not conic. But in general, we will not have separability and will have to resort to empirical numerical methods.

2 Actions define orbit families

Because actions restrict allowed motion to a subspace in phase space, the projection of the subspace onto coordinate space restricts trajectory in coordinate space to some well-define region.

For fully regular systems, there is a unique correspondence between the shape of the coordinate space filled by the trajectory, the gravitational potential and the orbit's conserved quantities. This is the basis for orbit classification.

For irregular system, some orbits will appear to fill the entire region allowed by energy conservation, some will appear to fill much smaller regions for quite a while, indicating the existence of approximate conserved quantities.

2.1 Two simple examples

These will give you a walk through of the basic machinery using simple systems with familiar solutions.

2.1.1 A free particle

The Hamiltonian of the particle in free space is:

$$H = \frac{p^2}{2}.$$

In other words, the Hamiltonian is simply the kinetic energy, written here in terms of the total momentum $\mathbf{p}$. Let's take this to be one-dimensional to start and generalize to multiple dimensions in the end.

From this, we may write the Hamilton-Jacobi equation:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0.$$

Obviously the problem is time independent so we have:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 = E$$

or

$$\frac{\partial S}{\partial q} = \sqrt{2E} \equiv P = v = \text{constant}.$$

Recall that we can pick the new momentum P to be the constant of integration or any function of that constant. Here we pick it to be $\sqrt{2E}$ which, physically, is the velocity. Therefore, up to a constant, we have $S = qv$ and now using the rules for canonical transformation defined by the generating function from the last lecture we have:

$$\begin{aligned} p &= \frac{\partial S}{\partial q} = v \\ Q &= \frac{\partial S}{\partial P} = x \end{aligned}$$

This is rectilinear motion (the long hard way!). Note that we may do this problem in three dimensions by defining a separable generating function: $S(\mathbf{x}) = \sum_{i=1}^{3} S_i(x_i)$. Convince yourself that $S = \mathbf{q} \cdot \mathbf{v}$ and therefore $\mathbf{p} = \mathbf{v}$ and $\mathbf{Q} = \mathbf{x}$.

2.1.2 The harmonic oscillator

The Hamiltonian is now:

$$H = \frac{p^2}{2} + \frac{x^2}{2}.$$

From this, we may write the Hamilton-Jacobi equation directly for this time-independent system:

$$\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{x^2}{2} = E \equiv \frac{I^2}{2}.$$

Here, let us choose $P = I = \sqrt{2E}$. Again, we are free to choose whatever function of E we like but this make the expression for S below convenient. This equation may be rewritten as

$$\frac{\partial S}{\partial q} = \sqrt{I^2 - x^2}$$

whose solution is obviously

$$S(I,x) = \int^x dx \sqrt{I^2 - x^2}.$$

The upper limit reminds us that this is a function of x. The lower limit can be left arbitrary because $S(I,x)$ is only defined up to an arbitrary constant; adding a lower limit simply defines the constant of integration. Using the rules for canonical transformation we get:

$$Q = \frac{\partial S}{\partial I} = \int^x dx \frac{I}{\sqrt{I^2 - x^2}} = I \sin^{-1}(x/I) + Q_o$$

(where Q_o is the integration constant) or

$$x = I \sin\left[(Q - Q_o)/I\right].$$

Note that Q has unusual units for a coordinate! Recall from the last lecture that we are free to choose a momentum such that the coordinate is an angle which goes through 2π in one cycle of the oscillator's motion. To do this we need to pick the new momentum variable such that $P = 1/2\pi \oint dxp$. Substituting from the canonical solution of the H-J equation we have:

$$P = \frac{1}{2\pi}\oint dx \frac{\partial S}{\partial x} = \frac{1}{2\pi}\oint dx \sqrt{I^2 - x^2} = \frac{I^2}{2\pi}\int_0^{2\pi} dx \sqrt{1 - t^2} = \frac{I^2}{2}$$

So the action is $P = I^2/2$. It immediately follows $Q = \partial S/\partial P$ that $Q = I \sin^{-1}(x/I) + Q_o$ or $x = I \sin(Q - Q_o)$ as expected for an angle-like variable.

2.2 A galaxy disk

This is a good opportunity to exercise all of the machinery we have developed so far on a non-trivial example. Let us assume that the disk density and potential are axisymmetric and use polar coordinates: R, ϕ. The Hamiltonian is then

$$H(\mathbf{p}, \mathbf{q}) = \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{2R^2}\right) + \Phi(R).$$

Substituting to get the H-J equation, we find

$$\frac{1}{2}\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{2R^2}\left(\frac{\partial S}{\partial \phi}\right)^2 + \Phi(R) = E.$$

By inspection, this equation separates if we write $S = S_R(R) + S_\phi(\phi)$ as follows:

$$R^2\left[\frac{1}{2}\left(\frac{\partial S}{\partial R}\right)^2 + \Phi(R) - E\right] = -\frac{1}{2}\left(\frac{\partial S}{\partial \phi}\right)^2.$$

By separation, we mean that each side of this equation must be constant because each side is a function of a different independent variable. If we take $S_\phi = h\phi$ (where the h is a constant whose meaning will become clear in a few moments), then

$$S_R = \int_{R_p}^{R} dR\,\left[2(E-\Phi(R)) - h^2/R^2\right]^{1/2}.$$

Note that it follows from conservation of energy that the integrand in square brackets is $\dot{R}$. The angular (ϕ) action is

$$I_\phi = \frac{1}{2\pi}\oint d\phi \frac{\partial S}{\partial \phi} = h$$

and the radial action is

$$I_R = \frac{1}{2\pi}\oint dR \frac{\partial S}{\partial R} = \frac{1}{2\pi}\oint dR\,\left[2(E-\Phi(R)) - h^2/R^2\right]^{1/2}.$$

I will leave derivation of the conjugate angles and frequencies as an exercise for those interested. You may have now all made the correct identification of h with angular momentum. Therefore, if the orbit is circular, $E = h^2/2R^2 + \Phi(R)$, and therefore $I_R = 0$. As the angular momentum h decreases for a fixed energy E, I_R increases.

The radial action is then indicates the quantity of radial motion, just as the name suggests.

For a given E and h, the equation $E = h^2/2R^2 + \Phi(R)$ admits two roots (for a bound orbit) which are the *turning points* of the orbit, the points where the radial velocity changes direction. The motion is then confined between two concentric circles. This sort of orbit is called a *loop orbit*.

2.3 Orbit families in elliptical and ellipsoidal coordinates

The axisymmetric planar disk is a simple example but all of the principles generalize. For example, because of the separability condition, the orbital turning points are constants in some coordinate. If we take a more complicated separable example, such as elliptical or ellipsoid coordinates[2], we find a wider range of orbital geometries, but the actions, angles, and frequencies may be derived similarly by turning the crank. A new kind of orbit appears in this geometry, a *box orbit*, which fills a coordinate region, but has no clear sense of rotation like a loop orbit.

3 What to do if H-J is not easily solved...

One of the most straightforward approaches is the *surface-of-section* constructed first proposed by Poincaré. The idea is a simple consequence of the isolating character of actions and the dimension of the resulting tori. I will illustrate this by means of example.

Take a two-dimensional potential: $U = U(x,y)$. The resulting phase space is four-dimensional: $(x, \dot{x}, y, \dot{y})$. Because the potential is

[2]Remember, all conic coordinate systems yield separable H-J solutions.

time-independent, the energy is conserved:

$$E = \text{constant} = \frac{\dot{x}^2 + \dot{y}^2}{2} + U(x, y).$$

This motion is on a three-dimensional surface and is still hard to visualize. Now take some planar slice through the three-dimensional surface. If the intersection of the orbit on our slice is a one-dimensional curve, the orbit must be a two-dimensional surface in the allowed phase space. But, if the intersection is a two-dimensional area, the orbit must fill the entire three-dimensional space.

In practice, we may proceed as follows. Using our numerical integrator, we plot the values of x and $\dot{x}$ when $y = 0$ and $\dot{y} >$. (This last condition is not necessary often helps make the plot easier to interpret.)