Computing Our Universe

Cosmology VI – Growth of Structure

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1 LINEAR PERTURBATION THEORY

1.1 Perturbations

The early universe was very smooth on all scales but today we see definite fluctuations (inhomogeneities) in the matter density – greatly overdense regions such as galaxies and clusters and greatly underdense regions such as voids. How did the matter organize in this way? One natural mechanism, especially on large scales, is self organization by gravitational instability.

Slightly overdense regions (created by initial conditions or physical processes, e.g. inflation) attract more matter in a runaway process, while underdense regions expand faster and become even more diffuse.

1.2 Gravitational Instability

Consider a sphere of matter of radius r and density $\bar{\rho} + \delta\rho$, where $\bar{\rho}$ is the unperturbed background average. By Newton's second law a mass point at r feels an acceleration

$$\ddot{r} = -(4\pi/3)G\,\delta\rho\,r.$$

Construct a shell neighboring the sphere with the space between evacuated, so the mass contained by each is the same. Choose its radius R such that the average interior density is $\bar{\rho}$.

An outside observer will not see the outer sphere as gravitationally different from the rest of the universe, so R would be constant in time in Newton's cosmology and $R \sim a(t)$ in Friedmann cosmology. The motion of the inner sphere is

$$r(t) = R(t)\,[1 - \epsilon(t)],$$

and the mass constraint $(\bar{\rho} + \delta\rho)r^3 = \bar{\rho}R^3$ implies that the density contrast $\delta(t) \equiv \delta\rho/\rho = 3\epsilon(t)$. Substituting into the acceleration equation gives a growth equation

$$\ddot{\delta}(t) = 4\pi\bar{\rho}\delta \quad \rightarrow \quad \delta(t) \sim e^{\sqrt{4\pi\bar{\rho}}\,t}$$

in Newton's cosmology, and in Friedmann cosmology

$$\ddot{\delta}(t) + 2(\dot{a}/a)\dot{\delta} = 4\pi\bar{\rho}\delta.$$

The new term entering is called the **Hubble drag**, having the form of a friction opposing clumping. The Newtonian exponential growth of an infinitesimal density perturbation is called **Jeans instability**. We will see soon that the drag from Hubble expansion reduces the exponential growth to power law growth.

Three common, more rigorous derivations are

- Liouville approach (phase space moments of Vlasov's equation)
- Fluid dynamics approach (Euler's equations)
- General relativity approach (Einstein-Bianchi equations)

They add one term corresponding to pressure, which was ignored in our sphere and shell construction. The final density perturbation equation is

$$\ddot{\delta}(t) + 2(\dot{a}/a)\dot{\delta} - c_s^2 a^{-2}\nabla^2\delta = 4\pi\bar{\rho}\delta,$$

where $c_s = \sqrt{\partial p/\partial \rho}$ is the sound speed in the matter "fluid".

1.3 Jeans Length and Mass

The Hubble drag and pressure terms both oppose clumping (growth of density) while the source term on the RHS encourages it. Which wins out? The relative magnitudes of the three terms are roughly $H^2 : c_s^2\lambda^{-2} : H^2$, where λ is the wavelength of the perturbation under consideration (the equation is linear so we can consider each wave mode separately, leading later to Fourier analysis).

There are two distinct cases, if $\lambda < c_s H^{-1}$ or not. This critical wavelength is the **Jeans length**

$$\lambda_J = c_s\sqrt{\pi/\bar{\rho}},$$

interpreted physically as the sound crossing scale (i.e. limit on bulk movement) in a dynamical timescale. Note for an ideal gas $c_s^2 = kT/m$ so $\lambda_J = (\pi kT/m\bar{\rho})^{1/2}$.

The **Jeans mass** is the mass contained in a region of diameter λ_J:

$$M_J = (\pi/6)\bar{\rho}\lambda_J^3,$$

and is crucial for structure formation.

Before recombination	$c_s^2 = 1/3$	$\lambda_J \approx 2500$ Mpc	$M_J \approx 10^{20.5} M_\odot$
After recombination	$c_s^2 = 5kT/3m$	$\lambda_J \approx 10$ kpc	$M_J \approx 10^5 M_\odot$

If $\lambda < \lambda_J$ ($M < M_J$) then no growth can occur. In the next section we assume $\lambda > \lambda_J$ and analyze the growth ignoring the Jeans damping.

2 GROWTH OF STRUCTURE

2.1 Solutions

In terms of cosmological parameters, the growth equation is

$$\ddot{\delta} + 2H\dot{\delta} - (3/2)H^2\Omega_0(t)\delta = 0.$$

When $\Omega = \Omega_0 = 1$ (flat matter universe) the solution is obvious. Using $H = (2/3)t^{-1}$ one finds

$$\delta = c_1 t^{2/3} + c_2 t^{-1}.$$

The first term grows with time; the second is the decaying mode. There is always one of each and we ignore the decaying mode – it does not grow structure.

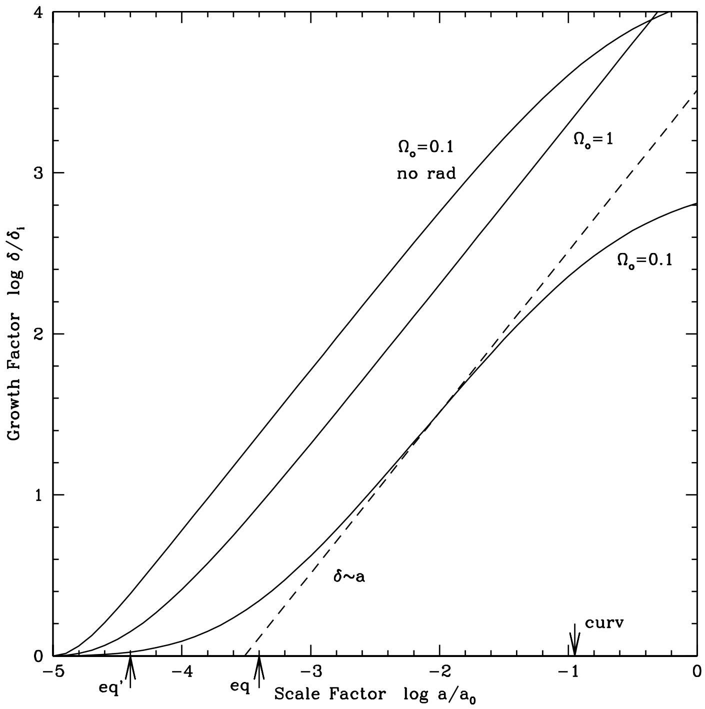
In terms of scale factor, we have the famous linear growth law:

$$\delta \sim a, \qquad (\Omega_0 = 1)$$

Since $\Omega_0 \to 1$ in the past (but still in the matter dominated era), this linear growth is valid for a long period even if today $\Omega_0 \neq 1$.

Summary of Density Growth

- Matter dominated ($\Omega_0 = 1$): $\delta \sim a$
- Low density: $\delta \sim D(t)$ with **growth factor** $D(t)$ slower than $a(t)$
- Curvature dominated: $\delta \sim$ const – no growth
- Radiation dominated: $\delta \sim$ const – no growth
- Cosmological constant: little effect so in $\Omega_m + \Omega_\Lambda = 1$ universe growth is like $\Omega_m = 1$



2.2 Categories of Perturbations

Modes:

- Scalar – density, pressure perturbations
- Vector – velocity, rotation perturbations (decay)
- Tensor – gravitational waves

Types:

- Adiabatic – $(\delta n/n)_b = (\delta n/n)_\gamma \Rightarrow (\delta\rho/\rho)_b = 3(\delta T/T)_\gamma$
- Isocurvature – $\delta\rho_b = -\delta\rho_\gamma$

2.3 Dark Matter

No density perturbation can grow in the radiation dominated era because the expansion rate (Hubble drag) is too large. When the matter dominated era begins, matter not interacting electromagnetically with photons

– (nonbaryonic) **dark matter** – can proceed with gravitational instability and cluster; baryons must wait until z_{dec}. These particles are collisionless (no pressure) so $M_J = 0$ and all scales can grow, but not equally. Key characteristic is their velocity at z_{eq}.

• Relativistic ($m <$ few eV): Hot dark matter (**HDM**), e.g. massive (but light) neutrinos

• Nonrelativistic ($m \gg eV$): Cold dark matter (**CDM**; e.g. WIMPs – hypothetical exotic particles like axion or lightest supersymmetric particle)

The primordial distribution with scale (**power spectrum**) is altered, characterized by the **transfer function** in wave number

$$T(k) = \frac{\delta_k(0)}{\delta_k(z_i)} \frac{D(z_i)}{D(0)}$$

• HDM free streams out of potential wells, erasing structure on scales up to $\lambda_{fs} \approx 40\,(30\,eV/m_\nu)$ Mpc. Structure forms **top-down**.

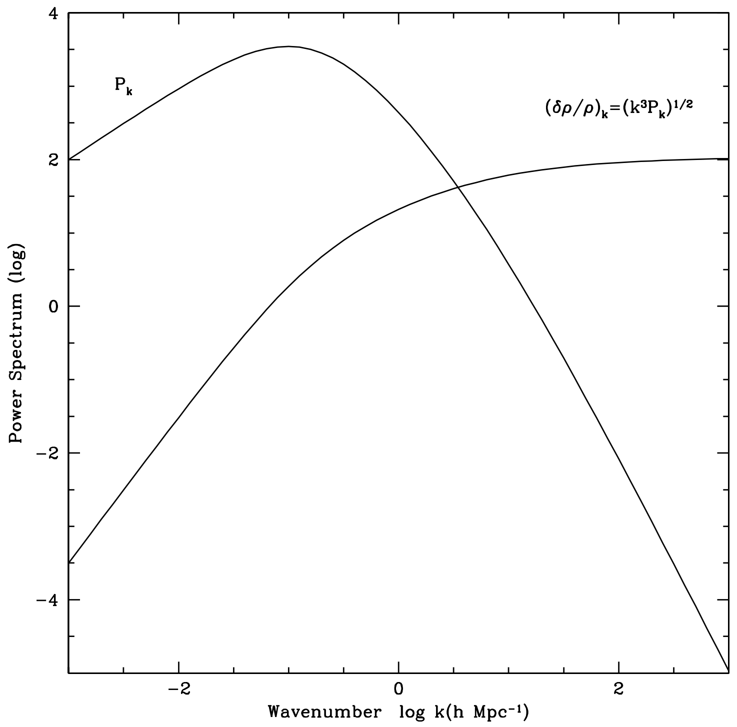
• CDM wavelengths inside/outside the horizon at equality, $d_{eq} = 16$ $(\Omega_0 h^2)^{-1}$ Mpc, give $T_k \sim k^{-2}$ on small scales. Structure forms **bottom-up**.

3 STATISTICAL DESCRIPTION OF STRUCTURE

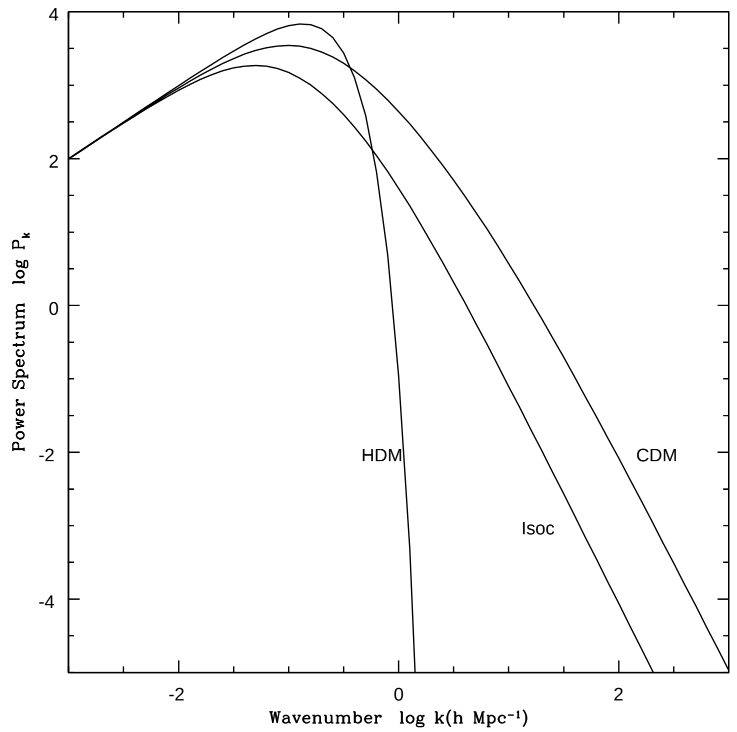
3.1 Correlation Functions

The two point **correlation function**, also good for describing nonlinear structure ($\delta > 1$), is

$$\xi(r) = \langle \delta(\vec{x})\, \delta(\vec{x}') \rangle_{r=|\vec{x}-\vec{x}'|}$$



P_k
(δρ/ρ)_k=(k³P_k)^1/2
Power Spectrum (log)
Wavenumber log k(h Mpc⁻¹)
4
2
0
−2
−4
−2
0
2



HDM
Isoc
CDM
Power Spectrum log P_k
Wavenumber log k(h Mpc⁻¹)
4
2
0
-2
-4
-2
0
2

Observationally it has the form

$$\xi(r) = (r/r_0)^{-\gamma}, \qquad \gamma \approx 1.8$$

If the three point and higher order correlations can be written as products of the two point one, the clustering is called **hierarchical**.

If the correlation function of galaxies equals the correlation function of mass density, one says *light traces mass*; otherwise there is **biasing**.

3.2 Power Spectrum

The Fourier transform of $\xi(r)$ is the **power spectrum** and gives the distribution in wave number (inverse wavelength) of the (squared) density fluctuation amplitudes.

$$\delta_k = (2\pi)^{-3/2} \int d^3x \, \delta(\vec{x}) \, e^{i\vec{k}\cdot\vec{x}}$$

$$\langle \delta_{\mathbf{k}} \delta_{\mathbf{k}'} \rangle = \langle |\delta_{\mathbf{k}}|^2 \rangle \, \delta^{(3)}(\mathbf{k} - \mathbf{k}') \equiv P_k \, \delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

$$P_k = (2\pi)^{-3} \int d^3r \, \xi(r) \, e^{-i\vec{k}\cdot\vec{r}}$$

If the perturbations are **gaussian** distributed (as inflation predicts)

$$f(\delta_{\mathbf{k}}) = (2\pi P_k)^{-3/2} e^{-|\delta_k|^2/2P_k},$$

then the distribution is determined solely by P_k.

Often one adopts a power law dependence

$$P_k \sim k^n,$$

with n the **spectral index**, hence

$$\langle (\delta\rho/\rho)^2 \rangle_\lambda \sim k^3 |\delta_k|^2 \sim \lambda^{-(3+n)}$$

$$(\Delta\phi)_\lambda \sim \lambda^{(1-n)/2}.$$

Note $n = 1$ (as inflation predicts) gives **scale invariant** (i.e. constant) potential perturbations.