*Computing Our Universe*

**Cosmology II – Dynamics of the Universe**

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# 1 FRIEDMANN EQUATIONS

## 1.1 Newtonian Approach

Homogeneity and isotropy make things very simple, quasiNewtonian.

"Matter evolving in closed system" recalls thermodynamics.

$a(t)$ – changing scales, i.e. volume, is like 1st law. Adiabatic $\Rightarrow$

$$dU = d(\rho V) = -p\,dV$$
$$\dot{\rho} = -(\rho + p)\frac{dV}{V}$$

Motion of particle: 1st derivative gives energy conservation

$$\frac{1}{2}m\dot{R}^2 + G\frac{mM}{R} = \text{constant}$$
$$\left(\frac{\dot{R}}{R}\right)^2 = -\frac{8\pi}{3}\rho + \text{constant}R^{-2}$$

2nd derivative gives Newton's 2nd law

$$m\ddot{R} = \frac{GMm}{R^2}$$
$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3}\rho$$

$R \sim a(t)$, $V \sim a^3$, and GR includes pressure in gravity:

**Friedmann Equations**

$$
\begin{aligned}
\dot{\rho} &= -3\frac{\dot{a}}{a}(\rho + p) \\
\left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi}{3}\rho - ka^{-2} \\
\frac{\ddot{a}}{a} &= -\frac{4\pi}{3}(\rho + 3p)
\end{aligned}
$$

## 1.2 General Relativity

GR well tested, gives rigorous derivation of Friedmann eqs.

3 unknowns ($a$, $\rho$, $p$) so need 3 eqs, but conservation eq redundant (used energy conservation twice/only 2 eq from hom and iso).

To close system, need equation of state from particle physics.

## 1.3 Equation of State

All components of interest follow EOS

$$p = w\,\rho$$

for example, matter ($w = 0$: "dust"), radiation ($w = 1/3$: photons, neutrinos, GW, relativistic matter), cosmological constant ($w = -1$).

## Review of Principles

Equivalence Principle

$\Rightarrow$ Metric Description

Homogeneity and Isotropy

$\Rightarrow$ Metric is Robertson-Walker

$\Rightarrow$ Energy-Momentum Tensor is in perfect fluid form

Gravitational Field Equations (general relativity)

& Homogeneity and Isotropy

$\Rightarrow$ Friedmann Equations for evolution of spacetime

Equation of State

& Friedmann Equations

$\Rightarrow$ Evolution of matter

# 2 SOLUTIONS

## 2.1 Density Evolution

Solution to conservation Friedmann eq, each component separately,

$$\rho_w \sim a^{-3(1+w)}$$

No interactions $\Rightarrow$ conserved separately; $w$, $N_w$ constant

EOS intuitive: $\rho_w \sim E_w N_w / V \sim E_w N_w a^{-3}$

**Figure 2.1: Illustration of Energy Density Evolution**

At any one time, one component tends to dominate. But each evolves at different rate so can be overtaken.

*Example:* Radiation $\rightarrow$ Matter. Since $\rho_{\mathrm{rad}} \sim a^{-4}$, $\rho_{\mathrm{mat}} \sim a^{-3}$ then

$$\rho_{\mathrm{rad}} > \rho_{\mathrm{mat}} \qquad \text{when} \qquad a/a_0 < \left(\frac{\rho_{\mathrm{rad}}}{\rho_{\mathrm{mat}}}\right)_{\mathrm{today}}$$

## 2.2 Time Evolution

Since LHS of velocity Friedmann eq is positive, there are different behaviors if $k = 0, \pm 1$, with $k = 0$ critical divider.

$$\begin{aligned} H &\equiv \dot{a}/a && \text{(Hubble parameter)} \\ \rho_c &\equiv \frac{3H^2}{8\pi} && \text{(critical density)} \\ \Omega &\equiv \rho/\rho_c && \text{(dimensionless density)} \end{aligned}$$

*Convention*: if no time argument, then evaluate at today, *except* $H(\text{today}) \equiv H_0$ – **Hubble constant**.

Can rewrite velocity eq as

$$k = [\Omega(a) - 1]\, H^2 a^2 = (\Omega - 1)\, H_0^2 = \text{constant}$$

Formal solution for $a(t)$ by inverting

$$t_0 - t = \int_a^{a_0} da/\dot{a} = \int_a^1 da\, a^{-1} \left[ (8\pi/3) \sum_w \rho_w\, a^{-3(1+w)} - k a^{-2} \right]^{-1/2}$$

When one component dominates (also over curvature) then

$$a \sim t^{\frac{2}{3(1+w)}}$$

Note $a(t)$ monotonic and currently observed to be increasing (expanding) so look back to past state of $a \to 0$ – **Big Bang**. Parametrize by time, or *redshift* $z = (a/a_0)^{-1} - 1$ running from 0 today to $\infty$ at Big Bang.

Age of universe is

$$t_0 \approx \frac{2}{3(1+w)} H_0^{-1}$$

Can measure $H_0^{-1}$ and compare: radioactive elements $0.9 \times 10^{10}$ y, oldest stars $1.3 \times 10^{10}$ y, $H_0^{-1} = 10^{10} h^{-1}$ y.

## Components' EOS and Evolution

| | | | |
|---|---|---|---|
| Radiation | $\rho_{1/3} \sim a^{-4}$ | $a \sim t^{1/2}$ | $t_0 = (1/2) H_0^{-1}$ |
| Nonrelativistic matter | $\rho_0 \sim a^{-3}$ | $a \sim t^{2/3}$ | $t_0 = (2/3) H_0^{-1}$ |
| Curvature | $\rho_{-1/3} \sim a^{-2}$ | $a \sim t$ | $t_0 = H_0^{-1}$ |
| Cosmological constant | $\rho_{-1} \sim a^0$ | $a \sim e^{H_0 t}$ | $t_0 = \infty$ |

### 2.3 Radiation-Matter Equality

Most of its age, universe is matter dominated. Early on radiation dominated, very recently may be curvature or cosmological constant dominated.

Epoch of transition from radiation $\rightarrow$ matter domination very important for structure formation.

$$\begin{aligned}
\rho_{1/3}(a) &< \rho_0(a) \\
\rho_{1/3}(a_0) \cdot (a/a_0)^{-4} &< \rho_0(a_0) \cdot (a/a_0)^{-3} \\
a &> a_0\, \rho_{1/3}(a_0)/\rho_0(a_0) = \Omega_{1/3}/\Omega_0 \\
1 + z \equiv a_0/a &< 23900\, \Omega_0 h^2
\end{aligned}$$

using that the radiation component is made of blackbody photons at $T = 2.73$ K and three species of light neutrinos, for $\rho_{1/3} = 7.85 \times 10^{-34}$ g cm$^{-3}$ or $\Omega_{1/3} = 4.18 \times 10^{-5} h^{-2}$.

Note that critical density is

$$\begin{aligned}
\rho_c &= 1.88 \times 10^{-29} \text{g cm}^{-3} \\
&= 2.78 \times 10^{11}\, M_\odot\, \text{Mpc}^{-3}
\end{aligned}$$

Matter-radiation equality, $z_{eq}$, is much earlier than when the first objects (galaxies and stars) form, at $z \approx 5$.

In terms of time, $t_{eq} \approx 10000$ y, so the universe was rather young at that epoch. At $z = 5$, the universe was about 8% its current age. Neither redshift nor time are linear in the sense that interesting things happen equally in equal intervals.