SPH

WHAT IS IT?

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References:

Hernquist, L. & Katz, N. 1989 *Ap. J. Suppl.* **70** 419.

Katz, N., Weinberg, D.H., & Hernquist, L. 1996 *Ap. J. Suppl.* **105** 19.

Why Include Gas?

- Gas has different dynamics than collisionless material.
    - Gas flows cannot interpenetrate. Hence gas moves on closed orbits.
    - Gas pressure is isotropic and hence gas cannot have anisotropic support.
    - Gas can shock converting bulk kinetic energy into thermal energy.
- Gas can heat or cool through radiative processes.
- We observe gas!

The Fluid Equations

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}$$

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla P}{\rho} - \nabla \phi$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u = -\left(\frac{P}{\rho}\right) \nabla \cdot \mathbf{v} + \frac{\Gamma - \Lambda}{\rho}$$

$$P = (\gamma - 1)\rho u$$

$$\nabla^2 \phi = 4\pi G \rho$$

Eulerian Techniques

- The standard approach for solving the fluid equations.
- Calculate the fluid properties on a regular grid of points.
- Calculate derivatives using finite differences.
- Several short comings for astrophysical problems.
  - Spatial resolution and dynamic range is limited, *e.g.* cosmological simulations.
  - Computational effort is often wasted on cells that are not of interest, *e.g.* colliding stars and polar ring evolution.
- Several strengths for astrophysical problems.
  - Can capture shocks in two grid cells by solving small versions of the Riemann problem, e.g. the piecewise parabolic method (PPM).
  - Can model low density gas.

Basic Principles of Smoothed Particle Hydrodynamics (SPH)

- Model the fluid as a collection of fluid elements represented by N particles.

- Move the particles using Lagrangian forms of the fluid equations.

- Some thermodynamic properties are assigned to each particle, *e.g.* mass and temperature.

- Other thermodynamic properties of the fluid, *e.g.* ∇P, are estimated using local averages.

- Since the computational model consists of a finite number of fluid elements, local averages must be made over volumes of finite extent.

- Local averages are made using an interpolation method that allows any function to be expressed in terms of its values at a set of disordered points—the particles. This is the heart of SPH.

Interpolation

- The integral interpolant of any function $f(r)$ is

$$< f(\mathbf{r}) >= \int W(\mathbf{r}-\mathbf{r}'; h) f(\mathbf{r}') d\mathbf{r}'$$

where the integration is over all space.

- h is the smoothing length and specifies the extent of the averaging volume.

- W is the smoothing kernel which must have two properties

$$\int W(\mathbf{r}-\mathbf{r}'; h) d\mathbf{r} = 1$$

and

$$\lim_{h\to 0} W(\mathbf{r}; h) = \delta(\mathbf{r}-\mathbf{r}')$$

.

- If the values of $f(r)$ are only known at a finite number of points then

$$< f(\mathbf{r}) >= \sum_{j=1}^{N} \frac{f(\mathbf{r}_j)}{< n(\mathbf{r}_j) >} W(\mathbf{r}-\mathbf{r}_j; h).$$

- In particular if a mass m_j is associated with each particle then

$$< \rho(\mathbf{r}) >= \sum_{j=1}^{N} m_j W(\mathbf{r} - \mathbf{r}_j; h)$$

and

$$< f(\mathbf{r}) >= \sum_{j=1}^{N} m_j \frac{f_j}{\rho_j} W(\mathbf{r} - \mathbf{r}_j; h).$$

- Gradients are calculated in a similar manner. By definition

$$< \nabla f(\mathbf{r}) >= \int W(\mathbf{r} - \mathbf{r}'; h) \nabla f(\mathbf{r}') d\mathbf{r}'.$$

Integrating by parts yields

$$< \nabla f(\mathbf{r}) >= \int f(\mathbf{r}') \nabla W(\mathbf{r} - \mathbf{r}'; h) d\mathbf{r}'.$$

- For a finite number of particles with mass m_j

$$< (\nabla f)_i >= \sum_{j=1}^{N} m_j \frac{f_j}{\rho_j} \nabla W(\mathbf{r} - \mathbf{r}_j; h).$$

- Higher accuracy is obtained by using the relation $\rho \nabla f = \nabla(\rho f) - f \nabla \rho$

giving

$$< (\nabla f)_i >= \frac{1}{\rho_i} \sum_{j=1}^{N} m_j (f_j - f_i) \nabla W(\mathbf{r} - \mathbf{r}_j; h).$$

The Momentum Equation

- In the Lagrangian form the momentum equation is

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla P}{\rho}.$$

- In particle form this equation is

$$\frac{(\nabla \mathbf{P})_i}{\rho_i} = \sum_j m_j (P_j - P_i) \nabla_i W(r_{ij}, h)$$

but this form is not symmetric in i and j so would not conserve linear and angular momentum.

- One can symmetrize the pressure gradient term using the relation

$$\frac{\nabla P}{\rho} = \nabla \left(\frac{P}{\rho} \right) + \frac{P}{\rho^2} \nabla \rho$$

to yield

$$\frac{(\nabla \mathbf{P})_i}{\rho_i} = \sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2} \right) \nabla_i W(r_{ij}, h).$$

If large gradients are present this form can give errors.

- An alternative symmetric form can be derived using $\nabla P = 2\sqrt{P}(\nabla \sqrt{P})$ then

$$\frac{(\nabla \mathbf{P})_i}{\rho_i} = \sum_j m_j \frac{\sqrt{P_i P_j}}{\rho_i \rho_j} \nabla_i W(r_{ij}, h).$$

This form is less sensitive to large gradients.

The Thermal Energy Equation

- In Lagrangian form the thermal energy equation is

$$\frac{du}{dt} = -\left(\frac{P}{\rho}\right)\nabla \cdot \mathbf{v} + \frac{\Gamma - \Lambda}{\rho}.$$

- Ignoring the heating and cooling terms this can be written in particle form as

$$\frac{du_i}{dt} = \frac{P_i}{\rho_i^2}\sum_{j=1}^{N} m_j \mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h).$$

- Another form can be obtained by noting

$$\frac{du}{dt} = -\nabla\left(\frac{P\mathbf{v}}{\rho}\right) + \mathbf{v} \cdot \nabla\left(\frac{P}{\rho}\right)$$

then

$$\frac{du_i}{dt} = \sum_{j=1}^{N} m_j \frac{P_j}{\rho_j^2}\mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h).$$

- Averaging these two forms one gets

$$\frac{du_i}{dt} = \frac{1}{2}\sum_{j=1}^{N} m_j \left(\frac{P_j}{\rho_j^2} + \frac{P_i}{\rho_i^2}\right)\mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h).$$

similar to the arithmetic mean form for the momentum equation.

- One can also derive a form similar to the geometric mean form of the momentum equation

$$\frac{du_i}{dt} = \sum_{j=1}^{N} m_j \left(\frac{\sqrt{P_i P_j}}{\rho_i \rho_j} \right) \mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h).$$

The Continuity Equation

- Since SPH is Lagrangian the continuity equation is just replaced by the expression

$$\rho_i = \sum_{j=1}^{N} m_j W(r_{ij}; h).$$

Artificial Viscosity

- To allow shocks to form and to prevent the interpenetration of high Mach number flows it is necessary to introduce an artificial viscosity.

- Artificial viscosity is added to the momentum and thermal energy equations. For the geometric mean forms these equations become

$$\frac{d\mathbf{v_i}}{dt} = -\sum_j m_j \left(2\frac{\sqrt{P_i P_j}}{\rho_i \rho_j} + \Pi_{ij} \right) \nabla_i W(r_{ij}, h)$$

and

$$\frac{du_i}{dt} = \sum_{j=1}^{N} m_j \left(\frac{\sqrt{P_i P_j}}{\rho_i \rho_j} + \frac{1}{2}\Pi_{ij} \right) \mathbf{v}_{ij} \cdot \nabla_i W(r_{ij}, h) + \frac{\Gamma - \Lambda}{\rho}$$

where Π_{ij} is the artificial viscosity between particles i and j.

- The standard form for the artificial viscosity is

$$\Pi_{ij} = \frac{-\alpha \mu_{ij} \bar{c}_{ij} + \beta \mu_{ij}^2}{\bar{\rho}_{ij}}$$

where

$$\mu_{ij} = \begin{cases} \frac{\mathbf{v}_{ij} \cdot \mathbf{r}_{ij}}{h_{ij}(r_{ij}^2/h_{ij}^2 + \eta^2)} & \text{for} \quad \mathbf{v}_{ij} \cdot \mathbf{r}_{ij} < 0 \\ 0 & \text{for} \quad \mathbf{v}_{ij} \cdot \mathbf{r}_{ij} \geq 0 \end{cases} .$$

Here, $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$, $\bar{c}_{ij} = (c_i + c_j)/2$ is the average speed of sound of particles i and j, $h_{ij} = (h_i + h_j)/2$, and $\bar{\rho}_{ij} = (\rho_i + \rho_j)/2$.

- The linear term is similar to a bulk viscosity.
- The quadratic term is necessary to prevent interpenetration at high Mach number and is roughly equivalent to the Von Neumann-Richtmyer viscosity used in finite difference methods.
- This form of the artificial viscosity has several desirable features.
  - It is Galilean invariant.
  - It vanishes for rigid body rotation.
  - It conserves total linear and angular momentum.
- A disadvantage of this form of the artificial viscosity is that it can introduce considerable amounts of shear viscosity.

- Can use a form that uses a true bulk viscosity and a Von Neumann-Richtmyer viscosity.

$$\Pi_{ij} = \frac{q_i}{\rho_i^2} + \frac{q_j}{\rho_j^2},$$

where

$$q_i = \begin{cases} \alpha h_i \rho_i c_i |\nabla \cdot \mathbf{v}|_i + \beta h_i^2 \rho_i |\nabla \cdot \mathbf{v}|_i^2 & \text{for} \quad \nabla \cdot \mathbf{v} < 0 \\ 0 & \text{for} \quad \nabla \cdot \mathbf{v} \geq 0 \end{cases}$$

To eliminate particles contributing viscous *cooling*, set $\Pi_{ij} = 0$ if $\mathbf{v}_{ij} \cdot \mathbf{r}_{ij} > 0$; a decision which may, in some cases, add small amounts of shear viscosity.

- This form of the artificial viscosity greatly reduces the shear viscosity but gives a somewhat less accurate description of the flow near shocks.

- Both forms successfully resolve shocks over $\sim 4h$.

Integrating in Time

- Integrate the the positions and velocities as in a typical N-body code using leap frog.

- Now the timestep is also limited by the Courant condition to ensure stability,

$$\Delta t < \mathcal{C}\frac{h}{c_{sound}}$$

  where $\mathcal{C} \sim 0.3$ is the Courant number. When including artificial viscosity this becomes

$$\Delta t = \mathcal{C}\frac{h}{h|\nabla \cdot \mathbf{v}| + c_{sound} + 1.2(\alpha c_{sound} + \beta h|\nabla \cdot \mathbf{v}|)}.$$

- The thermal energy equation is integrated using a predictor-corrector scheme. When heating and cooling are included a semi-implicit predictor corrector scheme is used.

Varying Resolution

- In systems that change extent with time, to retain resolution it is possible to allow h to vary with time. For example, choose

$$h \propto < \rho >^{-1/3}$$

$$< \rho > = \frac{1}{n} \sum_{j=1}^{N} \rho_j .$$

- To further increase the dynamic range it is also possible to allow h to vary spatially.

- If one adopts the scatter interpretation of SPH then

$$< f(\mathbf{r}) > = \int W(\mathbf{r} - \mathbf{r}', h(\mathbf{r}')) f(\mathbf{r}') d\mathbf{r}'$$

  implying that the discrete estimate of the density is computed according to

$$< \rho(\mathbf{r}_i) > = \sum_{j=1}^{N} m_j W(r_{ij}, h_j).$$

- In the gather interpretation the smoothed estimates become

$$< f(\mathbf{r}) > = \int W(\mathbf{r} - \mathbf{r}', h(\mathbf{r})) f(\mathbf{r}') d\mathbf{r}'$$

and

$$< \rho(\mathbf{r}_i) >= \sum_{j=1}^{N} m_j W(r_{ij}, h_i).$$

- The estimates are still second order accurate.
- Neither form is symmetric in i and j so in the momentum equation neither linear nor angular momentum would be conserved.
- One approach is to symmetrize the dynamical equations by substituting $(h_i + h_j)/2$ for h_i or h_j, but leave the smoothed estimate of the density in asymmetric form.
- Another approach is to average the gather and scatter interpretations. Then

$$< f(\mathbf{r}) >= \int f(\mathbf{r}') \frac{1}{2} \left[W(\mathbf{r} - \mathbf{r}', h(\mathbf{r}')) + W(\mathbf{r} - \mathbf{r}', h(\mathbf{r})) \right] d\mathbf{r}'$$

and

$$< \rho(\mathbf{r}_i) >= \sum_{j=1}^{N} m_j \frac{1}{2} \left[W(r_{ij}, h_i) + W(r_{ij}, h_j) \right].$$

- This approach has the advantage that it is similar to the a form derived from a Lagrangian variational principle and one can easily control the number of neighbors and hence the local accuracy.